4507/6507 Software and Hardware Verification
LTL Model Checking

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These slides contain material from Denisa Diaconescu, Georg Struth and Traian Florin Șerbănuță

Overview

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The model checking algorithm in three steps

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Generalized Nondeterministic Büchi Automata (GNBA)

- Translation of LTL formulas to automata
- Product automata
- Emptiness decision problem

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We will see

- not only what needs to be done
- but also why it works – and we will give proofs for that

Introduction

Preliminaries: Atom-Set Traces

Recall: For a set of states S, a labeling function $L : S \to \mathcal{P}(Atoms)$, an infinite sequence of states $\pi = s_0 s_1 s_2 \ldots$ and a formula φ, we know what $\pi \models_L \varphi$ (π satisfies φ w.r.t. labeling L) means.

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- S, L and $\pi = s_0 s_1 s_2 \ldots$ infinite sequence of states in S
- S', L' and $\pi' = s'_0 s'_1 s'_2 \ldots$ infinite sequence of states in S'

then, assuming $L(s_i) = L'(s'_i)$ for all $i \geq 0$, we have

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In other words, it is only the infinite sequence of atom sets $L(s_0)L(s_1)L(s_2)\ldots$ that matters – we call this the atom-set trace of $\pi = s_0 s_1 s_2 \ldots$ through L.

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Example: If $L(s_0) = \{a, b\}$, $L(s_1) = \{a\}$ and $L(s_2) = \{b\}$, then the atom-set trace of $s_0 s_2 s_1 s_2 s_0 \ldots$ is $\{a, b\}\,\{b\}\,\{a\}\,\{b\}\,\{a, b\}\ldots$

The LTL Model Checking Problem

Let $\mathcal{M} = (S, \rightarrow, L)$ be an LTS, $s_0 \in S$, and φ an LTL formula.

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(Remember that this means: $\pi \models_L \varphi$ for all $\pi \in \mathit{Paths}_{s_0}(\mathcal{M})$, where we write $\mathit{Paths}_{s_0}(\mathcal{M})$ for the set of paths of $\mathcal{M}$ that start in s_0.)

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In-class exercise. Please discuss the flaws of the following argument: It is obvious that the LTL model checking problem has an algorithmic solution, because both the LTS and the formula are finite objects, so whether the LTS satisfies a formula in a given state should be decidable by simply applying the definition of satisfaction and doing an exhaustive check through the finite set of states.

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- The automaton has a notion of acceped word, where a word will be an infinite sequence $A_0 A_1 A_2 \ldots$ of atom sets: for all $i \geq 0$, $A_i \in \mathcal{P}(Atoms)$. The set of its accepted words forms its accepted language.

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- It has the property that for any formula ψ, set of states S, labeling function $L : S \to \mathcal{P}(Atoms)$ and infinite sequences of states π,

  $\pi \models_L \psi$ iff the atom-set trace of π through L is accepted by $\mathcal{A}ut_\psi$

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  ($\mathcal{A}ut_{\psi}$ accepts precisely the atom-set traces of sequences that satisfy ψ)
- Thus, $\mathcal{A}ut_{\neg\varphi}$ accepts precisely the atom-set traces of sequences that do not satisfy φ.

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This results in a product automaton $(\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi}$ whose accepted words are those coming from the paths π of $\mathcal{M}$ that start in s_0 and do not satisfy φ:

$\pi \in Paths_{s_0}(\mathcal{M})$ and $\pi \not\models_L \varphi$

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- If the accepted language is non-empty, we obtain $\pi \in Paths_{s_0}(\mathcal{M})$ such that $\pi \not\models_L \varphi$. So we conclude No, it is not the case that $\mathcal{M}, s_0 \models \varphi$.

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Step 1. Construct an automaton A_ψ for any formula ψ such that, for any set of states S, labeling $L : S \to \mathcal{P}(Atoms)$ and infinite sequences of states π:
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If so, conclude $\mathcal{M}, s \models \varphi$; otherwise conclude $\mathcal{M}, s \not\models \varphi$.

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How is the check done? Why are the conclusions correct?

GNBAs

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A word w is said to be accepted by $\mathcal{A}ut$ if it has an accepting run in $\mathcal{A}ut$.

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Given a word $w = x_0 x_1 x_2 \ldots$, a run for w is an infinite sequence of states $q_0 q_1 q_2 \ldots$ with $q_0 \in I$ that transit via its letters: $q_i \xrightarrow{x_i} q_{i+1}$ for each $i \geq 0$.

A run $q_0 q_1 q_2 \ldots$ for w is called accepting if it visits infinitely often each of the accepting sets: for all $F \in \mathcal{F}$, the set $\{i \geq 0 \mid q_i \in F\}$ is infinite.

A word w is said to be accepted by $\mathcal{A}ut$ if it has an accepting run in $\mathcal{A}ut$. The language accepted by $\mathcal{A}ut$ is the set of words accepted by $\mathcal{A}ut$.

GNBA – Example

$\mathcal{A}ut = (\Sigma, Q, I, \to, \mathcal{F})$ where:

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&\Sigma = \{x, y\} \qquad Q = \{q_0, q_1, q_2\} \\
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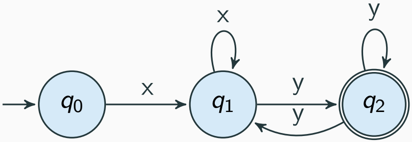

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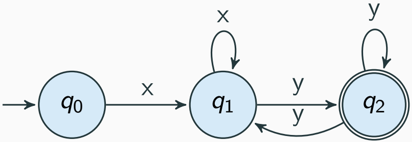

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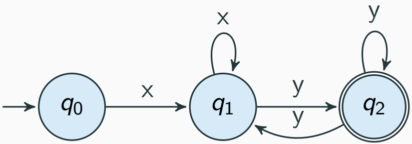

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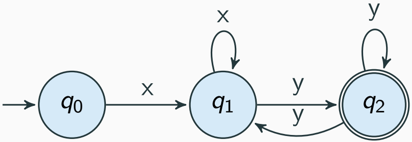

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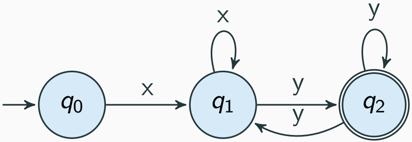

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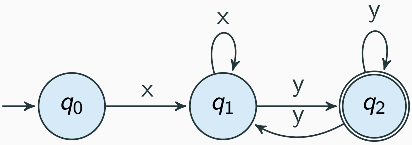

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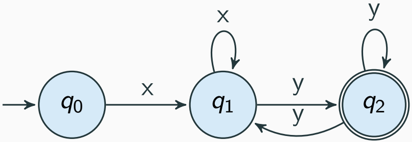

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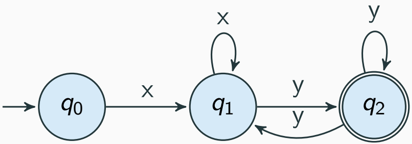

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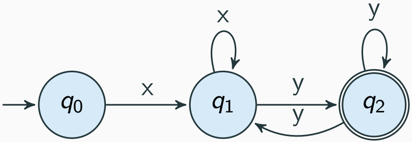

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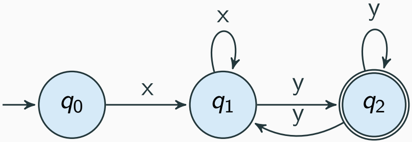

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$\mathrm{Lang}(\mathcal{A}ut)$ contains $x\,y^\infty$ and $x\,(x\,y^2)^\infty$, but not x^∞, $x\,y^2 x^\infty$, y^∞, $x\,y\,x^\infty$.

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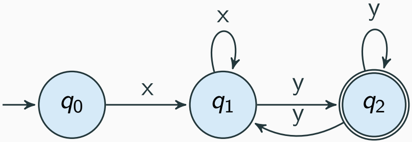

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Lang($\mathcal{A}ut$) contains $x\,y^\infty$ and $x\,(x\,y^2)^\infty$, but not x^∞, $x\,y^2 x^\infty$, y^∞, $x\,y\,x^\infty$.

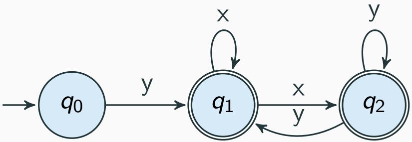
$$\begin{aligned}
\text{Lang}(\mathcal{A}ut) = &\{x^{m_1} y^{n_1} x^{m_2} y^{n_2} \ldots x^{m_p} y^{n_p} x^{m_{p+1}} y^\infty \mid p \geq 0, m_i > 0, n_i > 1\} \cup \\
&\{x^{m_1} y^{n_1} x^{m_2} y^{n_2} \ldots \mid m_i > 0, n_i > 1\}
\end{aligned}$$

Homework Exercise

Consider the following GNBA:

$\mathcal{A}ut = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ where:

$$\begin{array}{ll} \Sigma = \{x, y\} & Q = \{q_0, q_1, q_2\} \\ I = \{q_0\} & \mathcal{F} = \{\{q_1\},\ \{q_2\}\} \\ \multicolumn{2}{l}{\rightarrow = \{\,(q_0, y, q_1),\ (q_1, x, q_1),\ (q_1, x, q_2),} \\ \multicolumn{2}{l}{\qquad (q_2, y, q_1),\ (q_2, y, q_2)\,\}} \end{array}$$



Note: Here, a run is accepting iff <u>both</u> q_1 and q_2 appear in it infinitely often.

1. Which of the following words have runs, and which have accepting runs:
$y\,x^\infty$, $y\,x\,y^\infty$, x^∞, $(y\,x)^\infty$, $y\,(x^5\,y^3)^\infty$?
2. Can you describe the language accepted by $\mathcal{A}ut$?

Homework Exercise

Same questions as before, but for a slightly different GNBA – the only difference is shown in brown:

$\mathcal{A}ut = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ where:

$$\Sigma = \{x, y\} \qquad Q = \{q_0, q_1, q_2\}$$
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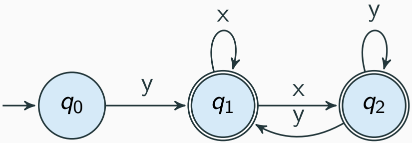

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We will often write $\overline{\varphi}$ instead of $\neg\varphi$ for any formula φ.

Important: We will identify (treat as if they are the same) $\overline{\overline{\varphi}}$ with φ – this is OK thanks to the Double Negation property.

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By the subformulas of ψ, we mean all the formulas that appear as part of ψ.

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An atom a has only one subformula: a itself.

$\Box a$ has two subformulas: a and $\Box a$.

$\Box\Diamond a$ has three subformulas: a, $\Diamond a$ and $\Box\Diamond a$.

$\Box(\overline{\Box a} \vee b)$ has six subformulas a, b, $\Box a$, $\overline{\Box a}$, $\overline{\Box a} \vee b$ and $\Box(\overline{\Box a} \vee b)$.

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The above is a "definition by example". Can you define the set of subformulas of a formula rigorously?

Step 1: From LTL Formulas to GNBAs – Discussion

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- we either demand a to be false

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For $\square a$ to be false:

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Thus, for the current state we have the following three possible scenarios:
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And we'll also have some requirements on moving forward to the next state.

Note that $\{\overline{a}, \Box a\}$ is not a possible scenario: It would be self-contradictory!

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\Diamond a$ will be true in the next state

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\Diamond a$ will be true in the next state

For $\Diamond a$ to be false:

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\lozenge a$. For $\lozenge a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\lozenge a$ will be true in the next state

For $\lozenge a$ to be false:

- we demand a to be false

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\Diamond a$ will be true in the next state

For $\Diamond a$ to be false:

- we demand a to be false
- we also demand that $\Diamond a$ will be false in the next state

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\Diamond a$ will be true in the next state

For $\Diamond a$ to be false:

- we demand a to be false
- we also demand that $\Diamond a$ will be false in the next state

Thus, for the current state we have the following three possible scenarios:
$\{a, \Diamond a\}$ $\{\overline{a}, \Diamond a\}$ $\{\overline{a}, \overline{\Diamond a}\}$

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\Diamond a$ will be true in the next state

For $\Diamond a$ to be false:

- we demand a to be false
- we also demand that $\Diamond a$ will be false in the next state

Thus, for the current state we have the following three possible scenarios:
$\{a, \Diamond a\}$ $\{\overline{a}, \Diamond a\}$ $\{\overline{a}, \overline{\Diamond a}\}$
And again, we'll have some requirements on moving forward to the next state.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
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- we demand a to be false
- we also demand that $\Diamond a$ will be false in the next state

Thus, for the current state we have the following three possible scenarios:
$\{a, \Diamond a\}$ $\{\overline{a}, \Diamond a\}$ $\{\overline{a}, \overline{\Diamond a}\}$
And again, we'll have some requirements on moving forward to the next state.

Note that $\{a, \overline{\Diamond a}\}$ is not a possible scenario: It would be self-contradictory.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\Diamond a$ will be true in the next state

For $\Diamond a$ to be false:

- we demand a to be false
- we also demand that $\Diamond a$ will be false in the next state

Thus, for the current state we have the following three possible scenarios:
$\{a, \Diamond a\}$ $\quad$ $\{\overline{a}, \Diamond a\}$ $\quad$ $\{\overline{a}, \overline{\Diamond a}\}$
And again, we'll have some requirements on moving forward to the next state.

Note that $\{a, \overline{\Diamond a}\}$ is not a possible scenario: It would be self-contradictory.

Homework Question: In which way is this similar to the discussion on the previous slide? Hint: $\Diamond$ and $\Box$ are dual to each other.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Diamond a$. For $\Diamond a$ to be true:

- we either demand a to be true
- or allow a to be false, but demand that $\Diamond a$ will be true in the next state

For $\Diamond a$ to be false:

- we demand a to be false
- we also demand that $\Diamond a$ will be false in the next state

Thus, for the current state we have the following three possible scenarios:
$\{a, \Diamond a\}$ $\{\bar{a}, \Diamond a\}$ $\{\bar{a}, \overline{\Diamond a}\}$
And again, we'll have some requirements on moving forward to the next state.

Note that $\{a, \overline{\Diamond a}\}$ is not a possible scenario: It would be self-contradictory.

Homework Question: In which way is this similar to the discussion on the previous slide? Hint: $\Diamond$ and $\Box$ are dual to each other.

In summary: We compute all possible scenarios for the correct state, and also remember some unfinished business for the next state.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box\Diamond a$. For $\Box\Diamond a$ to be true:

- we demand $\Diamond a$ to be true, hence:
  - we either demand a to be true
  - or allow a to be false, but demand that $\Diamond a$ will be true in the next state
- and demand $\Box\Diamond a$ to be true in the next state

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box\Diamond a$. For $\Box\Diamond a$ to be true:

- we demand $\Diamond a$ to be true, hence:
  - we either demand a to be true
  - or allow a to be false, but demand that $\Diamond a$ will be true in the next state
- and demand $\Box\Diamond a$ to be true in the next state

For $\Box\Diamond a$ to be false:

- we either demand $\Diamond a$ to be false, which also means that a is false
- or allow $\Diamond a$ to be true (but demand that $\Box\Diamond a$ will be false in the next state), in which case:
  - either a is true
  - or a is false but $\Diamond a$ will be true in the next state

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box\Diamond a$. For $\Box\Diamond a$ to be true:

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- we either demand $\Diamond a$ to be false, which also means that a is false
- or allow $\Diamond a$ to be true (but demand that $\Box\Diamond a$ will be false in the next state), in which case:
  - either a is true
  - or a is false but $\Diamond a$ will be true in the next state

Thus, for the current state we have five possible scenarios:

$\{a, \Diamond a, \Box\Diamond a\}$ $\{\bar{a}, \Diamond a, \Box\Diamond a\}$ $\{\bar{a}, \overline{\Diamond a}, \overline{\Box\Diamond a}\}$ $\{a, \Diamond a, \overline{\Box\Diamond a}\}$ $\{\bar{a}, \Diamond a, \overline{\Box\Diamond a}\}$

And we'll also have some requirements on moving forward to the next state.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box\Diamond a$. For $\Box\Diamond a$ to be true:

- we demand $\Diamond a$ to be true, hence:
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- we either demand $\Diamond a$ to be false, which also means that a is false
- or allow $\Diamond a$ to be true (but demand that $\Box\Diamond a$ will be false in the next state), in which case:
  - either a is true
  - or a is false but $\Diamond a$ will be true in the next state

Thus, for the current state we have five possible scenarios:

$\{a, \Diamond a, \Box\Diamond a\}$ $\{\overline{a}, \Diamond a, \Box\Diamond a\}$ $\{\overline{a}, \overline{\Diamond a}, \overline{\Box\Diamond a}\}$ $\{a, \Diamond a, \overline{\Box\Diamond a}\}$ $\{\overline{a}, \Diamond a, \overline{\Box\Diamond a}\}$

And we'll also have some requirements on moving forward to the next state.

Note that $\{\overline{a}, \overline{\Diamond a}, \Box\Diamond a\}$ is not a possible scenario. Why? Also, can you identify and explain other impossible scenarios?

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box(\overline{\Box a} \vee b)$.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box(\overline{\Box a} \vee b)$. For $\Box(\overline{\Box a} \vee b)$ to be true:

- we demand $\overline{\Box a} \vee b$ to be true, hence:
  1. we either demand $\overline{\Box a}$ to be true, in which case:
     1.1. we either demand a to be false
     1.2. or we allow it to be true, but demand that $\overline{\Box a}$ will be false in the next state
  2. or allow $\overline{\Box a}$ to be false, and demand b to be true
- we also demand that $\Box(\overline{\Box a} \vee b)$ will be true in the next state

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box(\overline{\Box a} \vee b)$. For $\Box(\overline{\Box a} \vee b)$ to be true:

- we demand $\overline{\Box a} \vee b$ to be true, hence:
  1. we either demand $\overline{\Box a}$ to be true, in which case:
     1.1. we either demand a to be false
     1.2. or we allow it to be true, but demand that $\overline{\Box a}$ will be false in the next state
  2. or allow $\overline{\Box a}$ to be false, and demand b to be true
- we also demand that $\Box(\overline{\Box a} \vee b)$ will be true in the next state

And a similar analysis yields all possibilities for $\Box(\overline{\Box a} \vee b)$ to be false.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box(\overline{\Box a} \vee b)$. For $\Box(\overline{\Box a} \vee b)$ to be true:

- we demand $\overline{\Box a} \vee b$ to be true, hence:
  1. we either demand $\overline{\Box a}$ to be true, in which case:
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     1.2. or we allow it to be true, but demand that $\overline{\Box a}$ will be false in the next state
  2. or allow $\overline{\Box a}$ to be false, and demand b to be true
- we also demand that $\Box(\overline{\Box a} \vee b)$ will be true in the next state

And a similar analysis yields all possibilities for $\Box(\overline{\Box a} \vee b)$ to be false.

Thus, for the current state we have the following possible scenarios:

1.1. $\{\bar{a},\ b,\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$ $\quad$ $\{\bar{a},\ \bar{b},\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$

1.2. $\{a,\ b,\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$ $\quad$ $\{a,\ \bar{b},\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$

2. $\{a,\ b,\ \Box a,\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$

... together with those for $\Box(\overline{\Box a} \vee b)$ to be false (not shown here).

And we'll also have some requirements on moving forward to the next state.

Step 1: From LTL Formulas to GNBAs – Discussion

Take ψ to be $\Box(\overline{\Box a} \vee b)$. For $\Box(\overline{\Box a} \vee b)$ to be true:

- we demand $\overline{\Box a} \vee b$ to be true, hence:
  1. we either demand $\overline{\Box a}$ to be true, in which case:
     1.1. we either demand a to be false
     1.2. or we allow it to be true, but demand that $\overline{\Box a}$ will be false in the next state
  2. or allow $\overline{\Box a}$ to be false, and demand b to be true
- we also demand that $\Box(\overline{\Box a} \vee b)$ will be true in the next state

And a similar analysis yields all possibilities for $\Box(\overline{\Box a} \vee b)$ to be false.

Thus, for the current state we have the following possible scenarios:

1.1. $\{\bar{a},\ b,\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$ $\quad$ $\{\bar{a},\ \bar{b},\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$

1.2. $\{a,\ b,\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$ $\quad$ $\{a,\ \bar{b},\ \overline{\Box a},\ \overline{\Box a} \vee b,\ \Box(\overline{\Box a} \vee b)\}$

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... together with those for $\Box(\overline{\Box a} \vee b)$ to be false (not shown here).

And we'll also have some requirements on moving forward to the next state.

Note. These scenarios are <u>complete</u>, i.e., answer the truth question on all subformulas, and <u>consistent</u>, i.e., they do not have contradictions, e.g., containing both φ and $\overline{\varphi}$, or containing $\Box\varphi$ but not φ.

Step 1: From LTL Formulas to GNBAs – Definition

$\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$

$\Sigma = \mathcal{P}(Atoms)$, so that words over this alphabet will be atom-set traces.

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We define $Cl(\psi)$, the closure of ψ, to be the set of all subformulas of ψ and their negations.

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$Cl\,(a) = \{a,\ \overline{a}\,\}$

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$Cl\,(\Box(\overline{\Box a} \vee b)) =$

$\{a,\, \overline{a},\, b,\, \overline{b},\, \Box a, \overline{\Box a},\, \overline{\Box a} \vee b,\, \overline{\overline{\Box a} \vee b},\, \Box(\overline{\Box a} \vee b),\, \overline{\Box(\overline{\Box a} \vee b)}\,\}$

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$Cl\,(\Box(\overline{\Box a} \vee b)) =$
$\{a,\ \overline{a},\ b,\ \overline{b},\ \Box a, \overline{\Box a},\ \overline{\Box a} \vee b,\ \overline{\overline{\Box a} \vee b},\ \Box(\overline{\Box a} \vee b),\ \overline{\Box(\overline{\Box a} \vee b)}\ \}$

$\overline{\overline{\Box a}}$ is not shown in $Cl\,(\Box(\overline{\Box a} \vee b))$, because it is the same as $\Box a$.

Step 1: From LTL Formulas to GNBAs – Definition

$\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ Recall: Q is the set of states of $\mathcal{A}ut_\psi$.

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- propositionally consistent means: for all $\varphi, \varphi_1, \varphi_2$
  - if $\varphi \in Cl(\psi)$, then $\varphi \in K$ implies $\overline{\varphi} \notin K$
  - if $\varphi_1 \wedge \varphi_2 \in Cl(\psi)$, then $\varphi_1 \wedge \varphi_2 \in K$ iff $\varphi_1 \in K$ and $\varphi_2 \in K$
  - if $\varphi_1 \vee \varphi_2 \in Cl(\psi)$, then $\varphi_1 \vee \varphi_2 \in K$ iff $\varphi_1 \in K$ or $\varphi_2 \in K$
  - if $\varphi_1 \to \varphi_2 \in Cl(\psi)$, then $\varphi_1 \to \varphi_2 \in K$ iff $[\varphi_1 \in K$ implies $\varphi_2 \in K]$

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  - if $\varphi_1 \vee \varphi_2 \in Cl(\psi)$, then $\varphi_1 \vee \varphi_2 \in K$ iff $\varphi_1 \in K$ or $\varphi_2 \in K$
  - if $\varphi_1 \to \varphi_2 \in Cl(\psi)$, then $\varphi_1 \to \varphi_2 \in K$ iff [$\varphi_1 \in K$ implies $\varphi_2 \in K$]

What do these conditions remind you of?

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  - if $\varphi_1 \rightarrow \varphi_2 \in Cl(\psi)$, then $\varphi_1 \rightarrow \varphi_2 \in K$ iff [$\varphi_1 \in K$ implies $\varphi_2 \in K$]

  What do these conditions remind you of?
- temporally consistent means:
  - if $\Box\varphi \in Cl(\psi)$, then $\Box\varphi \in K$ implies $\varphi \in K$
  - if $\Diamond\varphi \in Cl(\psi)$, then $\varphi \in K$ implies $\Diamond\varphi \in K$
  - if $\varphi_1 \,\mathsf{U}\, \varphi_2 \in Cl(\psi)$, then $\varphi_2 \in K$ implies $\varphi_1 \mathsf{U} \varphi_2 \in K$
  - if $\varphi_1 \,\mathsf{U}\, \varphi_2 \in Cl(\psi)$, then $\varphi_1 \,\mathsf{U}\, \varphi_2 \in K$ and $\varphi_2 \notin K$ implies $\varphi_1 \in K$

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  - if $\varphi_1 \,\mathsf{U}\, \varphi_2 \in Cl(\psi)$, then $\varphi_2 \in K$ implies $\varphi_1 \mathsf{U} \varphi_2 \in K$
  - if $\varphi_1 \,\mathsf{U}\, \varphi_2 \in Cl(\psi)$, then $\varphi_1 \,\mathsf{U}\, \varphi_2 \in K$ and $\varphi_2 \notin K$ implies $\varphi_1 \in K$

In short: Q consists of all the scenarios for the truth or falsehood of the subformulas of ψ that are complete (do not let anything unsettled) and consistent (do not contain contradictions).

Step 1: From LTL Formulas to GNBAs – Definition

$\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ Recall: I is the set of initial states of $\mathcal{A}ut_\psi$.

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We define $I =$ the set of all sets K in Q that contain ψ.

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So I incorporates all those scenarios where ψ is true.

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Intuition: The automaton will accept only the atom-set traces of sequences satisfying ψ, which therefore must start in scenarios where ψ is true.

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So I incorporates all those scenarios where ψ is true.

Intuition: The automaton will accept only the atom-set traces of sequences satisfying ψ, which therefore must start in scenarios where ψ is true.

Example, taking ψ to be $\Box\Diamond a$. Q consists of the following five sets, and I of only those that contain $\Box\Diamond a$ (the two ones shown in blue):

$\{a, \Diamond a, \Box\Diamond a\}$ $\{\bar{a}, \Diamond a, \Box\Diamond a\}$ $\{\bar{a}, \overline{\Diamond a}, \overline{\Box\Diamond a}\}$ $\{a, \Diamond a, \overline{\Box\Diamond a}\}$ $\{\bar{a}, \Diamond a, \overline{\Box\Diamond a}\}$

Step 1: From LTL Formulas to GNBAs – Definition

$\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ Recall: I is the set of initial states of $\mathcal{A}ut_\psi$.

We define $I =$ the set of all sets K in Q that contain ψ.

So I incorporates all those scenarios where ψ is true.

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In-class exercise: Please check this example against the definitions of Q and I.

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2. For all $\varphi, \varphi_1, \varphi_2$

- If $\bigcirc\varphi \in Cl(\psi)$, then $\bigcirc\varphi \in K$ iff $\varphi \in K'$
- If $\Box\varphi \in Cl(\psi)$, then $\Box\varphi \in K$ iff $\varphi \in K$ and $\Box\varphi \in K'$
- If $\Diamond\varphi \in Cl(\psi)$, then $\Diamond\varphi \in K$ iff $\varphi \in K$ or $\Diamond\varphi \in K'$
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  - $\Diamond\varphi$ true "now" means that either φ is true "now" or $\Diamond\varphi$ will be postponed to "next" (part of the "unfinished business")

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- So φ should be fulfilled in a "now" from the future $K'''^{\cdots\prime}$ after a finite chain of transitions $K \xrightarrow{A_1} K' \xrightarrow{A_2} K'' \rightarrow \ldots \rightarrow K''^{\cdots\prime}$.
- Thus, infinitely often, if $\Diamond\varphi$ is in the "now" then φ is also in the "now".

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And similarly for $\Box$ and U – they have their own long-term fulfillment goals.

Running Example

Let ψ be $\Diamond a$. Note that $Cl(\psi) = \{\, a,\ \overline{a},\ \Diamond a,\ \overline{\Diamond a}\,\}$

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Q consists of all elementary subsets of $Cl(\psi)$
Remember: Elementary = propositionally and temporally consistent, and complete
Thus: $Q = \{\ \{a,\, \Diamond a\},\ \{\overline{a},\, \Diamond a\},\ \{\overline{a},\, \overline{\Diamond a}\}\ \}$

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Thus: $I\ = \{\ \{a,\, \Diamond a\},\ \{\bar{a},\, \Diamond a\}\ \}$

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Let ψ be $\Diamond a$. Note that $Cl(\psi) = \{\, a,\, \overline{a},\, \Diamond a,\, \overline{\Diamond a}\,\}$

$\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ consists of the following:

$\Sigma = \mathcal{P}(\{a\}) = \{\, \emptyset,\, \{a\}\,\}$

Q consists of all elementary subsets of $Cl(\psi)$
Remember: Elementary = propositionally and temporally consistent, and complete
Thus: $Q = \{\ \{a,\, \Diamond a\},\ \{\overline{a},\, \Diamond a\},\ \{\overline{a},\, \overline{\Diamond a}\}\ \}$

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$\mathcal{F}$ contains one set, *Fulfill*$(\Diamond a)$.

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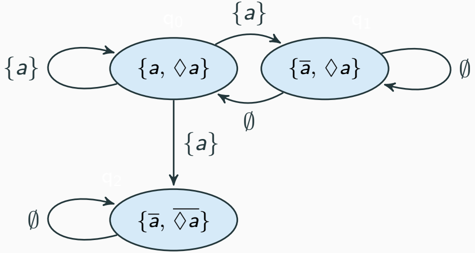
Note: Σ is a set of sets of atoms; Q, I and $Fulfill(\Diamond a)$ are sets of sets of formulas; $\mathcal{F}$ is a set of sets of sets of formulas.

The transition relation $\rightarrow$ is shown below:



Running Example

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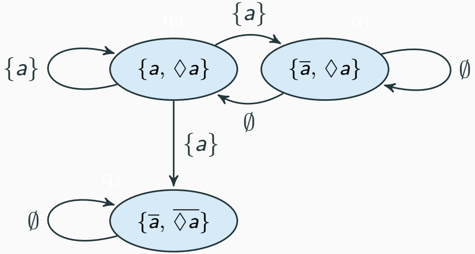

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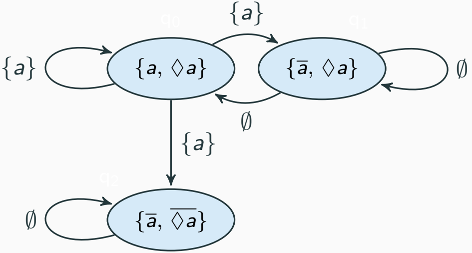

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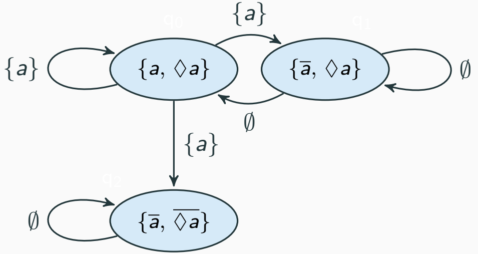

Note that for any $K, K' \in Q$ and $A \in \Sigma$, the conditions defining $K \xrightarrow{A} K'$ are:

1. A consists of all the atoms in K. This means that showing A is redundant, since it is determined by the source K of the transition:
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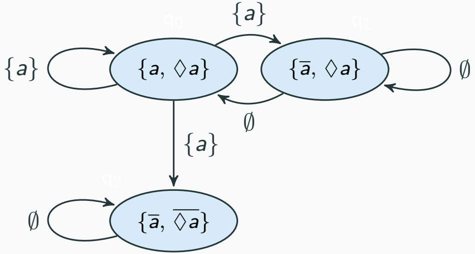

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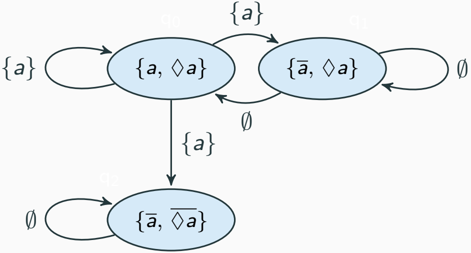

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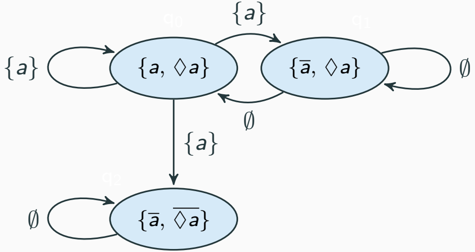

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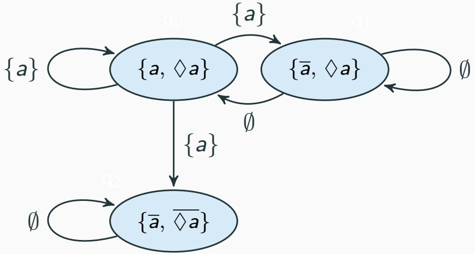

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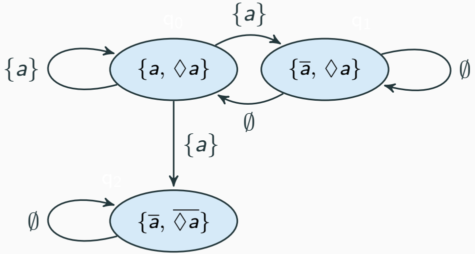

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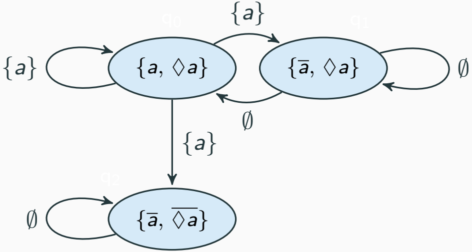

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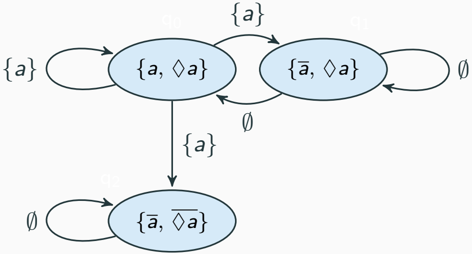

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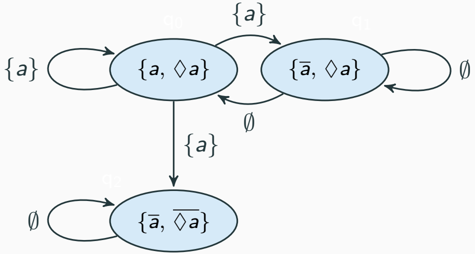
   - There is a transition between $\{a, \Diamond a\}$ and $\{\bar{a}, \Diamond a\}$, since this condition holds for $K = \{a, \Diamond a\}$ and $K' = \{\bar{a}, \Diamond a\}$
   - There is no transition between $\{\bar{a}, \overline{\Diamond a}\}$ and $\{\bar{a}, \Diamond a\}$, since this condition fails for $K = \{\bar{a}, \overline{\Diamond a}\}$ and $K' = \{\bar{a}, \Diamond a\}$ – indeed, $\Diamond a \in K'$ but $\Diamond a \notin K$

Running Example



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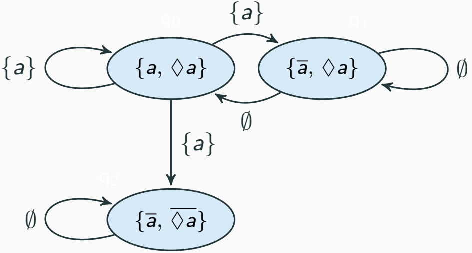
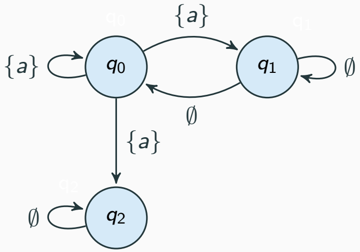


$I \;=\{\ \{a, \Diamond a\},\ \{\overline{a}, \Diamond a\}\ \}$

$\mathcal{F} = \{\mathit{Fulfill}(\Diamond a)\} = \{\ \{\ \{a, \Diamond a\},\ \{\overline{a}, \overline{\Diamond a}\}\ \}\ \}$

Running Example



$I = \{ q_0, q_1 \}$

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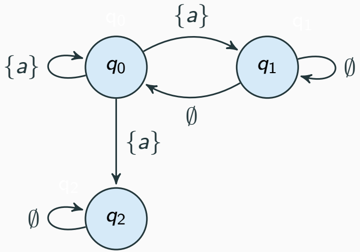
Running Example



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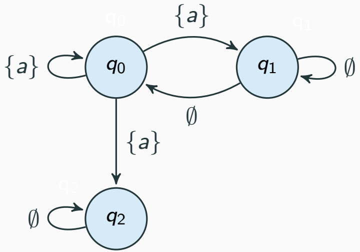
Accepted language of $\mathcal{A}ut_{\Diamond a}$?

Running Example



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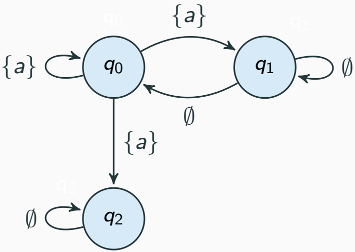

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... and this is exactly the property we need from the atom-set trace of a sequence π satisfying $\Diamond a$.

Step 1: From LTL Formulas to GNBAs – Correctness

Next, we will prove the following:

Correctness Theorem for Step 1. For any set of states S, infinite sequence of states π and labeling functions $L : S \to \mathcal{P}(Atom)$

$$\pi \models_L \psi \;\text{ iff }\; \mathcal{A}ut_\psi \text{ accepts the atom-set trace of } \pi \text{ through } L.$$

Step 1: From LTL Formulas to GNBAs – Correctness

Left-to-Right Implication of Correctness Theorem for Step 1. For any set of states S, infinite sequence of states π and labeling functions $L : S \to \mathcal{P}(Atom)$:

If $\pi \models_L \psi$ then $\mathcal{A}ut_\psi$ accepts the atom-set trace of π through L.

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Proof idea. Assume $\pi \models_L \psi$. Assume $\pi = s_0 s_1 s_2 \ldots$ and let $A_i = L(s_i)$ for all $i \geq 0$. We must show that $\mathcal{A}ut_\psi$ accepts $A_0 A_1 A_2 \ldots$, i.e., it has an accepting run for it.

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We can check that:

(1) $K_0 K_1 K_2 \ldots$ is a run, meaning:

- $K_i \in Q$, i.e., K_i is elementary – thanks to the properties of satisfaction
- $K_0 \in I$, i.e., $\psi \in K_0$ – immediate, since $\pi \models_L \psi$.
- $K_i \xrightarrow{A_i} K_{i+1}$ – thanks to the properties of satisfaction, incl. the expansion laws

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Step 1: From LTL Formulas to GNBAs – Correctness

Left-to-Right Implication of Correctness Theorem for Step 1. For any set of states S, infinite sequence of states π and labeling functions $L : S \to \mathcal{P}(Atom)$:

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Given $\Diamond\varphi \in Cl(\psi)$, we must check that $\Diamond\varphi \in K_i$, i.e., $\pi^i \models_L \Diamond\varphi$, implies $\varphi \in K_i$, i.e., $\pi^i \models_L \varphi$, for infinitely many i's.

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This is true because $\pi^i \models_L \Diamond\varphi$ implies that there exists $j \geq i$ such that $\pi^j \models_L \Diamond\varphi$ and $\pi^j \models_L \varphi$.

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This is true because $\pi^i \models_L \Diamond\varphi$ implies that there exists $j \geq i$ such that $\pi^j \models_L \Diamond\varphi$ and $\pi^j \models_L \varphi$. TYU: Prove this last statement.

Step 1: From LTL Formulas to GNBAs – Correctness

Right-to-Left Implication of Correctness Thm. for Step 1. For any S, π and L:

If $\mathcal{A}ut_\psi$ accepts the atom-set trace of π through L, then $\pi \models_L \psi$.

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Our generalization involves:

- replacing our fixed formula ψ with an arbitrary $\varphi \in K_0$
- renouncing the hypothesis (1.2) (of starting in an initial state)
- strengthening "$\varphi \in K_0$ implies $\pi \models_L \varphi$" to an "iff" statement, namely: (*) for all $\varphi \in Cl(\psi)$, we have $\varphi \in K_0$ iff $\pi \models_L \varphi$

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So we prove (1.1), (1.3) and (2) imply (*).

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So we prove (1.1), (1.3) and (2) imply (*). $\boxed{\text{TYU: Why is this more general?}}$

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iff (by the definition of $\mathcal{A}ut_\psi$'s transition relation $\rightarrow$)

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This was an easy case.

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Assume φ has the form $\varphi_1 \wedge \varphi_2$. We have a chain of equivalent statements:

$$\varphi_1 \wedge \varphi_2 \in K_0$$

iff (since K_0 is elementary, in particular propositionally consistent)

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iff (by the induction hypothesis)

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iff (by the induction hypothesis) TYU: OK to apply the induction hypothesis?

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iff (by the definition of the satisfaction relation)

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$$\pi \models_L \varphi_1 \text{ and } \pi \models_L \varphi_2$$

iff (by the definition of the satisfaction relation)

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This case is entirely routine; and the same is true for all propositional connectives.

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Assume φ has the form $\neg\varphi_1$. We have a chain of equivalent statements:

$$\neg\varphi_1 \in K_0$$

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Assume φ has the form $\neg\varphi_1$. We have a chain of equivalent statements:

$$\neg\varphi_1 \in K_0$$

iff (since K_0 is elementary, in particular propositionally consistent and complete)

$$\varphi_1 \notin K_0$$

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(1.1) K_i is elementary; (1.3) $K_i \xrightarrow{A_i} K_{i+1}$;

(2) $K_0 K_1 K_2 \dots$ visits infinitely often the sets in $\mathcal{F}$.

We must show: for all $\varphi \in Cl(\psi)$, we have $\varphi \in K_0$ iff $\pi \models_L \varphi$.

The proof goes by induction on the structure of φ. Some representative cases:

Assume φ has the form $\neg\varphi_1$. We have a chain of equivalent statements:

$$\neg\varphi_1 \in K_0$$

iff (since K_0 is elementary, in particular propositionally consistent and complete)

$$\varphi_1 \notin K_0$$

iff (by the induction hypothesis)

$$\pi \not\models_L \varphi_1$$

iff (by the definition of the satisfaction relation)

$$\pi \models_L \neg\varphi_1$$

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$$\neg\varphi_1 \in K_0$$

iff (since K_0 is elementary, in particular propositionally consistent and complete)

$$\varphi_1 \notin K_0$$

iff (by the induction hypothesis)

$$\pi \not\models_L \varphi_1$$

iff (by the definition of the satisfaction relation)

$$\pi \models_L \neg\varphi_1$$

This case is also entirely routine; but only because the statement to be proved is strong enough! An "implies" instead of "iff" would not work.

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Assume φ has the form $\bigcirc \varphi_1$. Let the atom-set trace $K'_0 K'_1 K'_2 \dots$ be defined as $K'_i = K_{i+1}$ for all $i \geq 0$. In other words, $K'_0 K'_1 K'_2 \dots$ is $K_1 K_2 K_3 \dots$

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The proof goes by induction on the structure of φ. Some representative cases:

Assume φ has the form $\bigcirc\varphi_1$. Let the atom-set trace $K_0' K_1' K_2' \ldots$ be defined as $K_i' = K_{i+1}$ for all $i \geq 0$. In other words, $K_0' K_1' K_2' \ldots$ is $K_1 K_2 K_3 \ldots$ We have:

$$\bigcirc\varphi_1 \in K_0$$

iff (by the definition of $\mathcal{A}ut_\psi$'s transition relation $\rightarrow$)

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iff (by the induction hypothesis applied to $K'_0 K'_1 K'_2 \dots$ and π^1)

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This case required applying the induction hypothesis not to $K_0 K_1 K_2 \ldots$ and π, but to their shifted versions $K_1 K_2 K_3 \ldots$ and π^1.

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This case required applying the induction hypothesis not to $K_0 K_1 K_2 \ldots$ and π, but to their j-shifted versions $K_j K_{j+1} K_{j+2} \ldots$ and π^j. CYA: Anything missing in this proof?

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Lemma: For all φ such that $\Diamond\varphi \in Cl(\psi)$, we have

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Proof idea.

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Since $K_0 \xrightarrow{A_0} K_1 \xrightarrow{A_1} K_2 \rightarrow \ldots$, from the definition of $\rightarrow$ we have that:
- Either (1) $\varphi \in K_0$ or (2) $[\varphi \notin K_0$ and $\Diamond\varphi \in K_1]$

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- And so on.

Moreover, $K_0 K_1 K_2 \ldots$ is an accepting run in $\mathcal{A}ut_\psi$, which means that infinitely often for $j \geq 0$, $\Diamond\varphi \in K_j$ implies $\varphi \in K_j$.

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Moreover, $K_0 K_1 K_2 \ldots$ is an accepting run in $\mathcal{A}ut_\psi$, which means that infinitely often for $j \geq 0$, $\Diamond\varphi \in K_j$ implies $\varphi \in K_j$.
So case (2) cannot hold infinitely, meaning that case (1) will hold at some point j.

Step 1: From LTL Formulas to GNBAs – Correctness

Lemma: For all φ such that $\Diamond\varphi \in Cl(\psi)$, we have

$$\Diamond\varphi \in K_0 \quad \text{iff} \quad \varphi \in K_j \text{ for some } j \geq 0$$

Proof idea.

For the left-to-right direction, assume $\Diamond\varphi \in K_0$.
Since $K_0 \xrightarrow{A_0} K_1 \xrightarrow{A_1} K_2 \to \ldots$, from the definition of $\to$ we have that:
- Either (1) $\varphi \in K_0$ or (2) $[\varphi \notin K_0$ and $\Diamond\varphi \in K_1]$

Case (1) means fulfilling the eventuality, whereas case (2) means postponing it to next time – remember the “unfinished business” situation.
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We thus obtain j such that $\varphi \in K_j$, as desired.

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So we obtain $\Diamond\varphi \in K_0$, as desired.

Homework Exercise

Assume $\pi = s_0 s_1 s_2 \ldots$ and let $A_i = L(s_i)$ for all $i \geq 0$. Assume:

(1.1) K_i is elementary; (1.3) $K_i \xrightarrow{A_i} K_{i+1}$;

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Do the proofs for the remaining cases:

Assume φ has the form $\varphi_1 \vee \varphi_2$. ... Routine

Assume φ has the form $\varphi_1 \rightarrow \varphi_2$. ... Routine

Assume φ has the form $\Box\varphi_1$. ... Interesting. You will need a lemma like for $\Diamond$.

Assume φ has the form $\varphi_1 \,\mathsf{U}\, \varphi_2$. ... Interesting. You will need a lemma like for $\Diamond$.

Summary and Outlook

For any formula ψ, we defined the GNBA $\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$.

We proved the following:

Correctness Theorem for Step 1. For any set of states S, infinite sequence of states π and labeling functions $L : S \rightarrow \mathcal{P}(Atom)$

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Finally, we will look into how to algorithmically decide satisfaction, once encoded – this is Step 3.

Homework Exercise

Describe the automaton $\mathcal{A}ut_\psi$ in the following cases:

- $Atoms = \{a\}$ and $\psi = \Box a$.
- $Atoms = \{a, b\}$ and $\psi = a \mathsf{U} b$
- $Atoms = \{a, b\}$ and $\psi = \Diamond(a \wedge b)$

Step 2: Product GNBA

Step 2: Product GNBA – Definition

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We define the product of $(\mathcal{M}, s_0)$ and $\mathcal{A}ut_\psi$ to be the GNBA $(\mathcal{M}, s_0) \times \mathcal{A}ut_\psi = (\Sigma, Q_\times, I_\times, \rightarrow_\times, \mathcal{F}_\times)$ whose components are as follows:

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Running Example (Continued)

Consider the LTS $\mathcal{M} = (S, \rightarrow, L)$ shown in the picture on the left.
Remember that, taking ψ to be $\Diamond a$, the GNBA $\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ has set of states Q and transition relation $\rightarrow$ shown in the picture on the right. Also,
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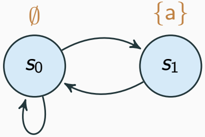

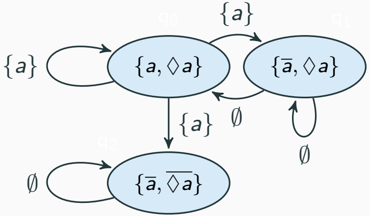

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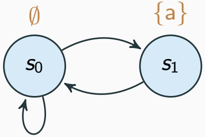

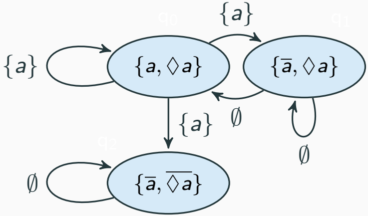

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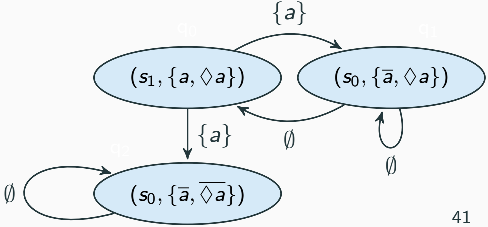

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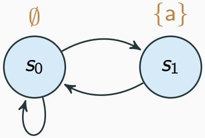

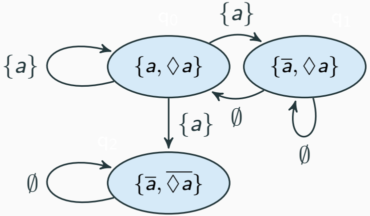

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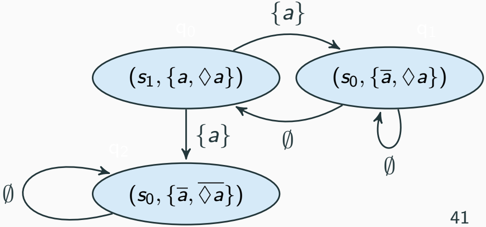

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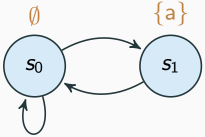

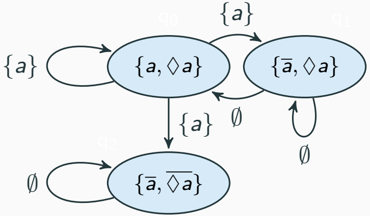

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and has $I_\times = \{\, (s_0, \{\overline{a}, \Diamond a\}) \,\}$ and
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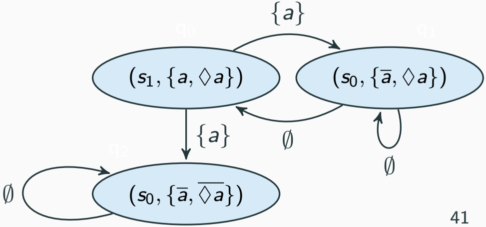
E.g., $Q_\times$ contains $(s_1, \{a, \Diamond a\})$
since $L(s_1) = \{a\} = \{a, \Diamond a\} \cap Atoms$



Running Example (Continued)

Consider the LTS $\mathcal{M} = (S, \to, L)$ shown in the picture on the left.
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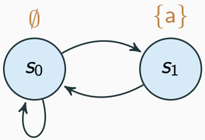

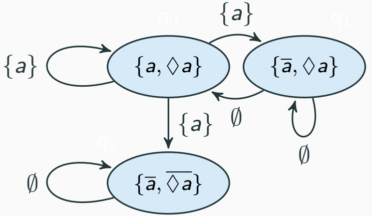

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E.g., $Q_\times$ does not contain $(s_1, \{\overline{a}, \Diamond a\})$
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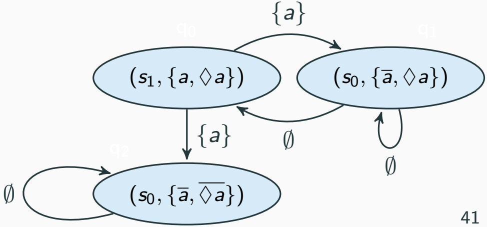

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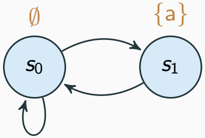

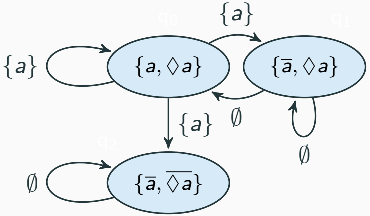

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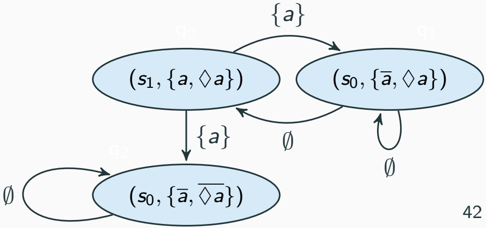

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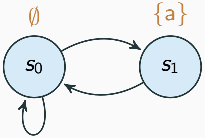

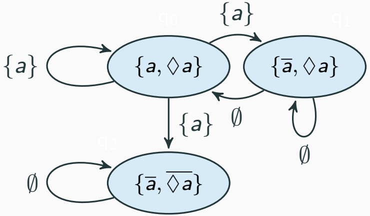

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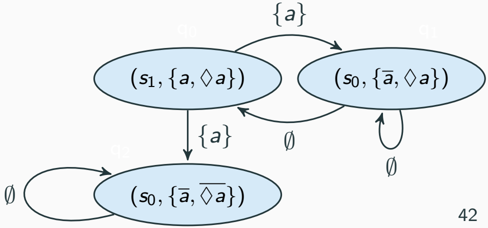

Step 2: Product GNBA – Correctness

Context: $\mathcal{M} = (S, \rightarrow, L)$ is an LTS, $s_0 \in S$, and $\mathcal{A}ut_\psi = (\Sigma, Q, I, \rightarrow, \mathcal{F})$ is the GNBA of an LTL formula ψ.

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Correctness Theorem for Step 2. Let $A_0 A_1 A_2 \ldots$ be an infinite sequence of atom sets. Then

$(\mathcal{M}, s_0) \times \mathcal{A}ut_\psi$ accepts $A_0 A_1 A_2 \ldots$
iff
there exists $\pi \in Paths_{s_0}(\mathcal{M})$ such that $A_0 A_1 A_2 \ldots$ is the atom-set trace of π through L and $\mathcal{A}ut_\psi$ accepts $A_0 A_1 A_2 \ldots$

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$$\text{atom-set trace of } \pi \text{ through } L \text{ and } \mathcal{A}ut_\psi \text{ accepts } A_0 A_1 A_2 \ldots$$

This has a routine proof, applying the definition of the product automaton.

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Proof. We have the following chain of equivalent statements:

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iff (by the definition of accepting runs and of $Q_\times$, $I_\times$ and $\rightarrow_\times$)

There exist $s_0 s_1 s_2 \ldots$ and $K_0 K_1 K_2 \ldots$ such that: $K_0 \in I$,
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iff (by the definition of accepting runs, of paths and of "atom-set trace of")

There exist $\pi = s_0 s_1 s_2 \ldots \in Paths_{s_0}(\mathcal{M})$ and $K_0 K_1 K_2 \ldots$ such that:
$A_0 A_1 A_2 \ldots$ is the atom-set trace of π through L and
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Corollary.

The language accepted by $(\mathcal{M}, s_0) \times \mathcal{A}ut_\psi$ is empty
iff
there exists no $\pi \in Paths_{s_0}(\mathcal{M})$ such that the atom-set trace of π through L is accepted by $\mathcal{A}ut_\psi$.

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The product between an LTS with a state and the GNBA of the negation of a formula encodes the satisfaction relation

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The language accepted by $(\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi}$ is empty
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iff (by the definition of satisfaction by LTSs)
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Hence, the Overall Correctness Theorem reduces the model checking problem for LTL, namely determining whether $\mathcal{M}, s_0 \models \varphi$, to the problem of determining whether the language accepted by the GNBA $(\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi}$ is empty.

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Our last piece in the puzzle:

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Our last piece in the puzzle:

Decidablity Theorem. Emptiness for GNBA is decidable, meaning: There is a program that takes as input a GNBA $\mathcal{A}ut$, always terminates, and returns

- 'Yes', if $\text{Lang}(\mathcal{A}ut) = \emptyset$
- 'No', if $\text{Lang}(\mathcal{A}ut) \neq \emptyset$

Step 3: Deciding Emptiness for GNBAs

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For any GNBA $\mathcal{A}ut = (\Sigma, Q, I, \rightarrow, \mathcal{F})$, we define its graph $Gr(\mathcal{A}ut) = (Q, \rightarrow)$ to be the following directed graph:

- The nodes of $Gr(\mathcal{A}ut)$ are the states Q
- Given $q_1, q_2 \in Q$, there is an edge between q_1 and q_2, written $q_1 \rightarrow q_2$, iff there exists a transition $q_1 \xrightarrow{x} q_2$ for some $x \in \Sigma$.

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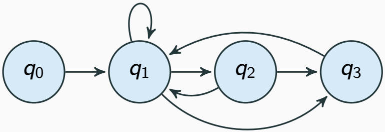

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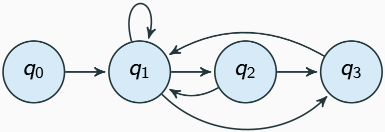

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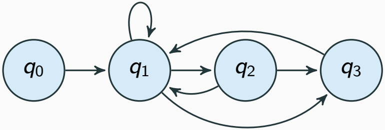

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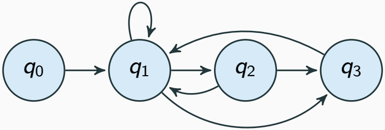

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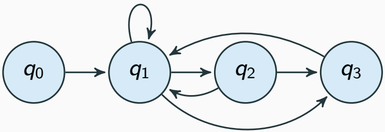

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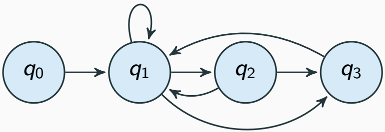

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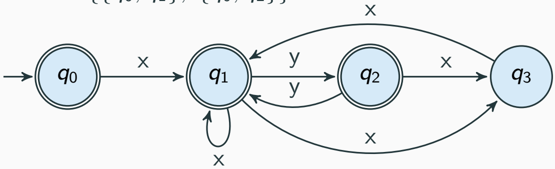

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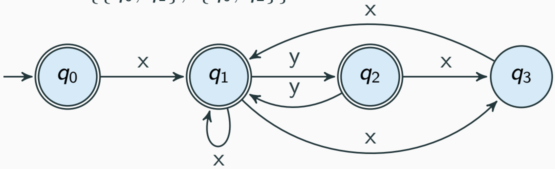

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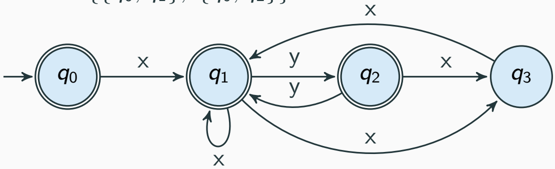

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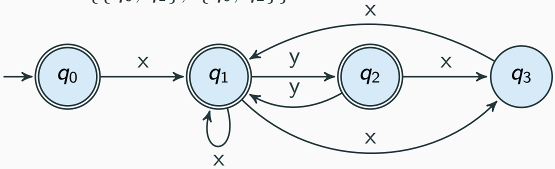

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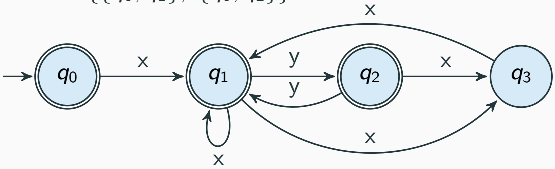

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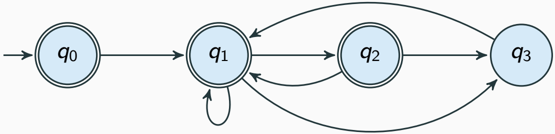

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Output "Yes, the accepted language is empty."

Summary of the LTL Model Checking Algorithm

Input: An LTS $\mathcal{M} = (S, \to, L)$, a state $s_0 \in S$, and an LTL formula φ.

Step 1: Compute the GNBA $\mathcal{A}ut = \mathcal{A}ut_{\neg\varphi}$.

Step 2: Compute the GNBA $\mathcal{A}ut' = (\mathcal{M}, s_0) \times \mathcal{A}ut$.

Step 3: Check whether Lang$(\mathcal{A}ut') = \emptyset$.

- If True, then output "Yes, it is the case that $\mathcal{M}, s_0 \models \varphi$."
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Running Example (Completed)

Let φ be $\neg \Diamond a$.

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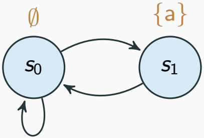

PROBLEM INSTANCE: $\boxed{\text{Does } \mathcal{M}, s_0 \models \varphi?}$

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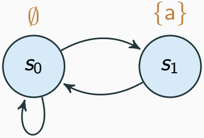
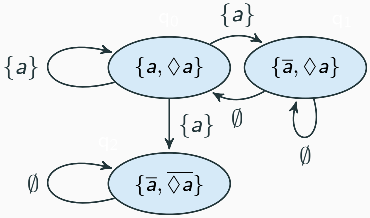


STEP 1: $\mathcal{A}ut_{\neg\varphi} = (\Sigma, Q, I, \rightarrow, \mathcal{F})$



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$$I = \{\, \{a, \Diamond a\},\ \{\overline{a}, \Diamond a\} \,\}$$
$$\mathcal{F} = \{\, \{\, \{a, \Diamond a\},\ \{\overline{a}, \overline{\Diamond a}\} \,\} \,\}$$

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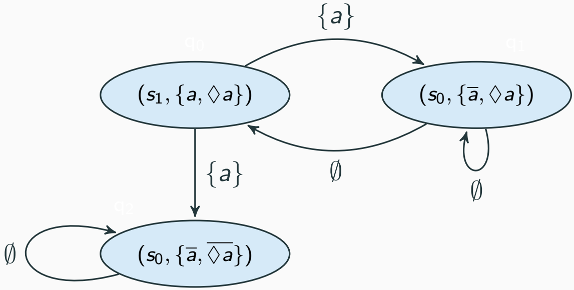
STEP 2: $(\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi} = (\Sigma, Q_\times, I_\times, \rightarrow_\times, \mathcal{F}_\times)$



$$I_\times = \{\, (s_0, \{\overline{a}, \Diamond a\}) \,\}$$
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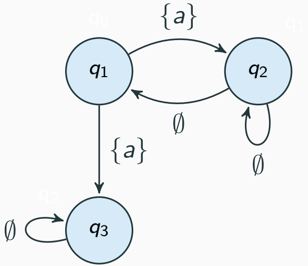

$I_\times = \{ q_2 \}$

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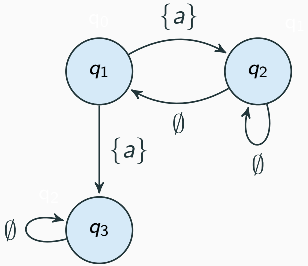

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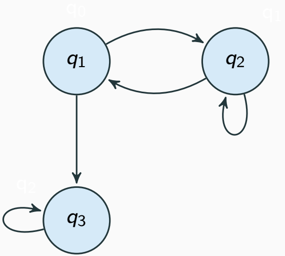
STEP 3: $Gr((\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi})$



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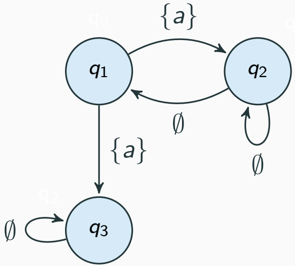

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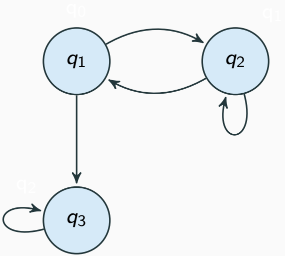

Two maximal non-trivial SCCs: $\{q_1, q_2\}$ and $\{q_3\}$.

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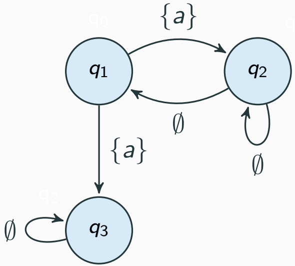

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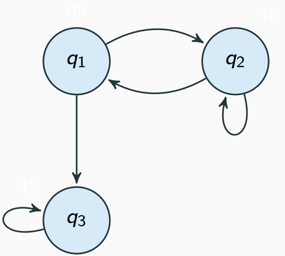

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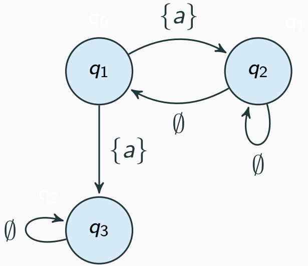

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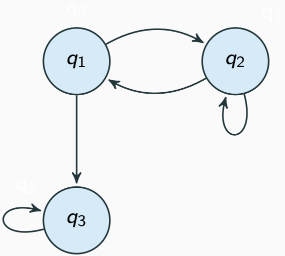

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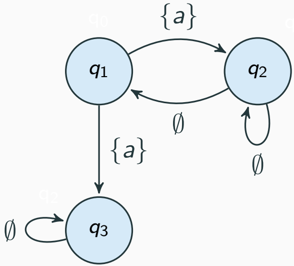

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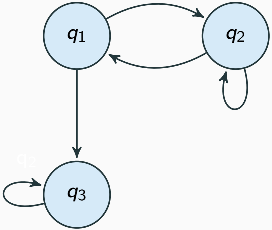

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Hence Lang$((\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi}) \neq \emptyset$.

We conclude: No, it is not the case that $\mathcal{M}, s_0 \models \varphi$.

Running Example (Completed)

Let φ be $\neg \Diamond a$. Then $\neg \varphi = \overline{\overline{\Diamond\ a}} = \Diamond\ a$. (Remember we identify $\overline{\overline{\varphi}}$ with φ.)

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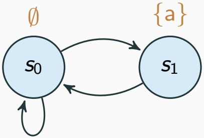

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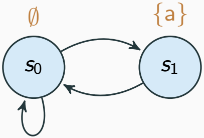


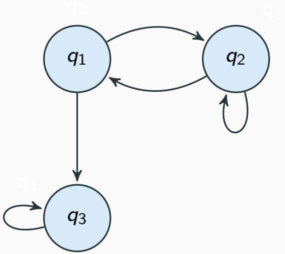
PROBLEM INSTANCE: $\boxed{\text{Does } \mathcal{M}, s_0 \models \varphi?}$

We conclude: No, it is not the case that $\mathcal{M}, s_0 \models \varphi$.

For example, $(s_0 s_1)^\infty \models \Diamond\ a$, hence $(s_0 s_1)^\infty \not\models_L \neg \Diamond\ a$, i.e., $(s_0 s_1)^\infty \not\models_L \varphi$.

Counterexample Path

$(\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi} = (\Sigma, Q_\times, I_\times, \rightarrow_\times, \mathcal{F}_\times)$ $\quad$ $Gr((\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi})$



$I_\times = \{q_2\}$

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Found $\{q_1, q_2\}$ maximal non-trivial SCC.

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$\{q_1, q_2\}$ intersects the only accepting set, $\{q_1, q_3\}$.

Hence Lang$((\mathcal{M}, s_0) \times \mathcal{A}ut_{\neg\varphi}) \neq \emptyset$. We conclude: $\mathcal{M}, s_0 \not\models \varphi$.

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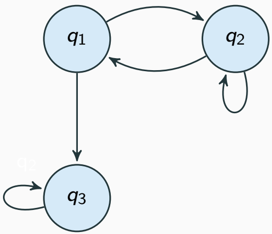

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Build a lasso: Start with a path from an initial state to our SCC: here, just q_2.

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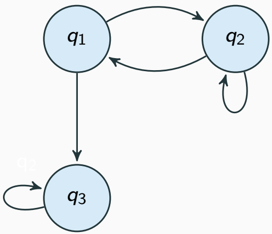

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Continue with a cycle that covers the entire SCC: $q_2 q_1 q_2$.

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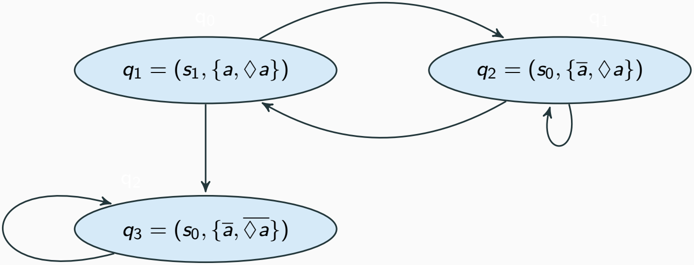

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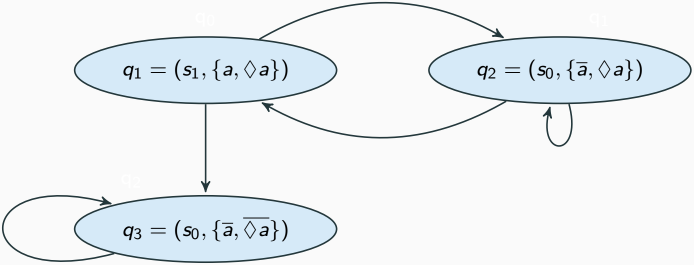

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Take the LTS state component of the product states: $s_0 s_1 s_0$.

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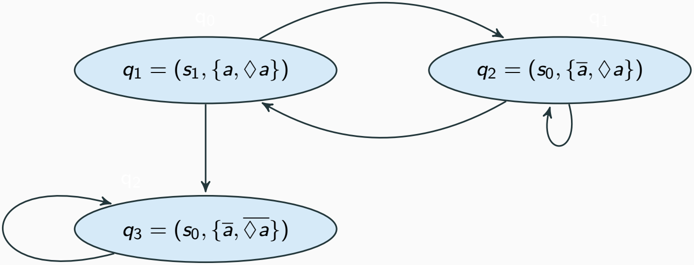

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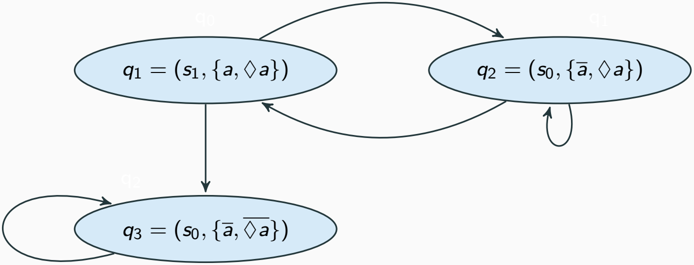

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This gives us a counterexample path: $(s_0 s_1)^\infty$. Indeed, $(s_0 s_1)^\infty \not\models_L \varphi$.

Complexity

Complexity of the LTL Model Checking Algorithm

Input: An LTS $\mathcal{M} = (S, \to, L)$, a state $s_0 \in S$, and an LTL formula φ.

Step 1: Compute the GNBA $\mathcal{A}ut = \mathcal{A}ut_{\neg\varphi}$.

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Can be done in $O(|\mathcal{A}ut'|)$ time and space.

Overall complexity: $O(|\mathcal{M}| \times 2^{O(|\varphi|)})$ time and space.

Summary

Summary of the Discussed Concepts

The model checking problem for LTL

GNBA = Generalized Nondeterministic Büchi Automata

Language accepted by a GNBA

Translation of LTL formulas to GNBAs

Construction of product GNBAs

Deciding the emptiness for (the language accpted by) GNBAs

The three steps of the LTL model checking algorithm

Time and space complexity

Possible Group Presentation Topic

In groups of three, implement the LTL model checking algorithm, where each member of the group takes care of one of the three steps.

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Some coordination is of course necessary, but the three steps can be coupled quite loosely if you agree on their input and output formats.

Feel free to discuss on the COM4507/6507 forum your choice of programming language, libraries, data structures, etc.

Note: This task would also be a good preparation for the exam!

Further Reading

Section 5.2 of Baier & Katoen's "Principles of Model Checking" (MIT Press 2008)

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Section 5.2 of Baier & Katoen's "Principles of Model Checking" (MIT Press 2008)

Moshe Vardi. An automata-theoretic approach to linear temporal logic. 1996.

Moshe Vardi. Automata-Theoretic Model Checking Revisited. 2007.