Supernominal Datatypes and Codatatypes

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University of Sheffield, UK

LFMTP
June 30, 2020

Those Were the Days...

LFMTP 2009, Montreal, Canada. Theory support for weak higher order abstract syntax in Isabelle/HOL. Elsa Gunter, Chris Osborn and Andrei Popescu.

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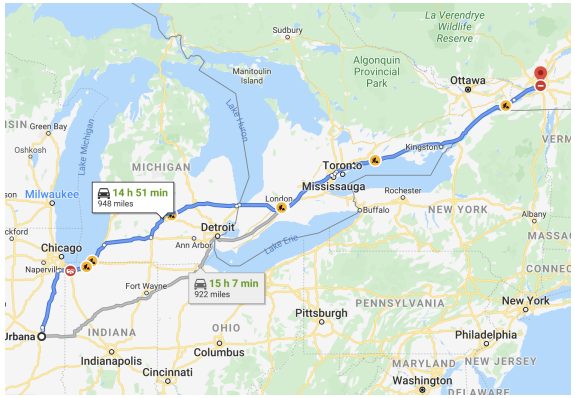
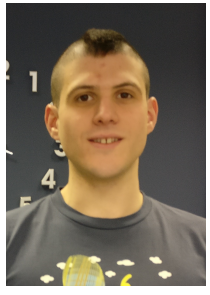
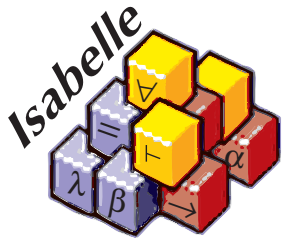


Supernominal

Joint work with...

Jasmin Blanchette, Lorenzo Gheri, Dmitriy Traytel, Isabelle/HOL





Ideology

How do most mathematicians, logicians and computer scientists view syntax with bindings?

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Syntax with Bindings

=

Bureaucracy

Domain-Specific Results

=

Interesting Bits

How do most mathematicians, logicians and computer scientists view syntax with bindings?

Syntax with Bindings

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Bureaucracy

Domain-Specific Results

=

Interesting Bits

Keep buraucracy low, offer high-level definition and proof principles.

Different Schools of Thought / Paradigms / Dogmas

De Bruijn

HOAS (Higher-Order Abstract Syntax)

Nominal/Nameful

Different Schools of Thought / Paradigms / Dogmas

De Bruijn nameless pointers

HOAS (Higher-Order Abstract Syntax) (meta-level) functional bindings

Nominal/Nameful explicit bound names, α-quotienting

They define terms with bindings in different ways.

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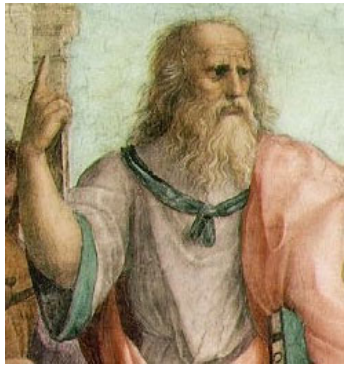

They talk about the same datatype

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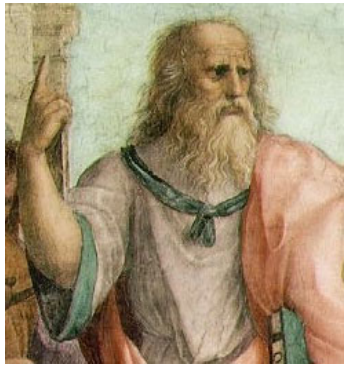

They talk about the same datatype
... but offer different ways to manipulate it

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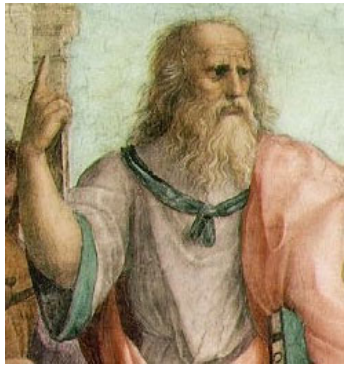

They talk about the same datatype
... but offer different ways to manipulate it

What's important: How to manipulate this datatype.

Different Schools of Thought / Paradigms / Dogmas

De Bruijn nameless pointers

$\lambda : \mathrm{Term} \to \mathrm{Term}$ $\lambda_n : \mathrm{Term}_{n+1} \to \mathrm{Term}_n$ (well-scoped)

HOAS (Higher-Order Abstract Syntax) (meta-level) functional bindings

$\lambda : (\mathrm{Term} \to \mathrm{Term}) \to \mathrm{Term}$ (strong) $\lambda : (\mathrm{Var} \to \mathrm{Term}) \to \mathrm{Term}$ (weak)

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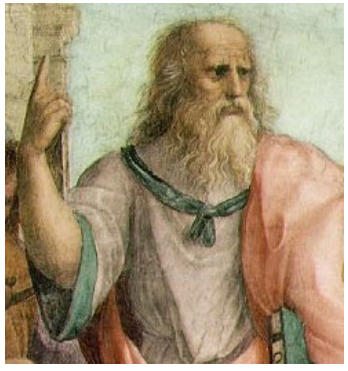

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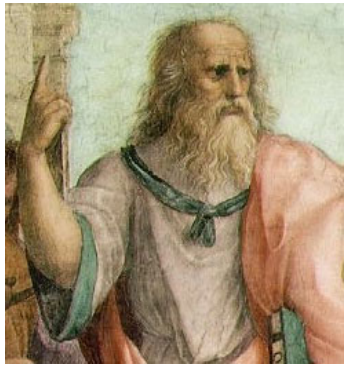

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What's important: How to manipulate this datatype.

Different schools have different insights and can learn from each other.

Combining and Sharing Knowledge across Paradigms

Johan van der
Brunomhoas
1471–1530

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Discovers a weak HOAS recursor

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Adjusts the above into a Nominal-style recursor

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Compares Nominal and "raw" De Bruijn

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Next: Paradigm-agnostic description of general binders
Nominal-style reasoning infrastructure for them
Could employ De Bruijn or HOAS views of the same datatypes!

Supernominal 

Theory of binding-aware datatypes and codatatypes

Supports modular specification of complex binding mechanisms

(Co)datatypes come with definition and reasoning principles

Generalizes Nominal techniques: no finite support restriction

Has been formalized in Isabelle (implementation under way)

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Then can have any syntax with any bindings we want

... which does not mean we lose any visual intuition

Bounded Natural Functors (BNFs)

Container Intuition: Shape Filled In With Content

$$\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$$

$$\mathsf{F}\,(A)$$

$$\biguplus$$

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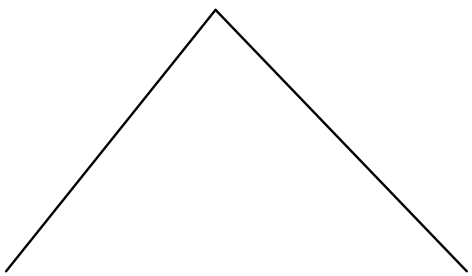
$$\Psi$$



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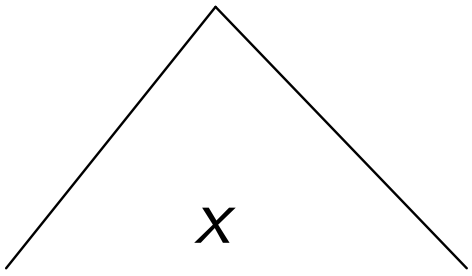
Ψ



where $x \in A$

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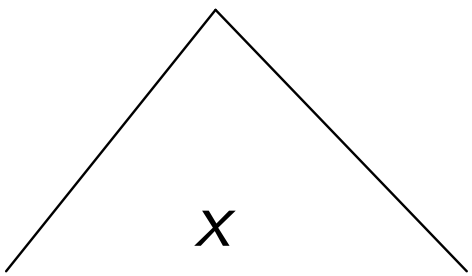

where $x \in A$

Example

$\mathsf{F}\,(A) = \mathbb{N} \times A$

$\mathsf{F}\,(A)$

Ψ

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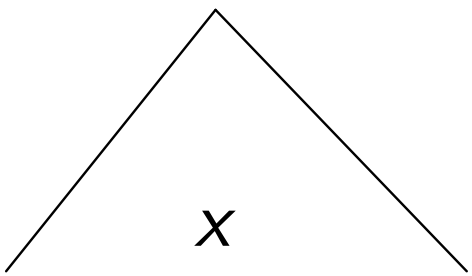
$\Downarrow$



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$(n, \_)$

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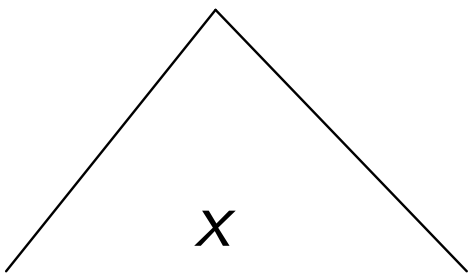

where $x \in A$

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$\mathsf{F}\,(A) = \mathbb{N} \times A$

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Ψ

(n, x)

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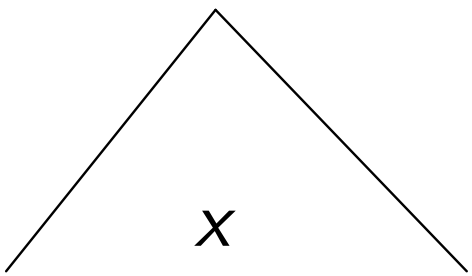

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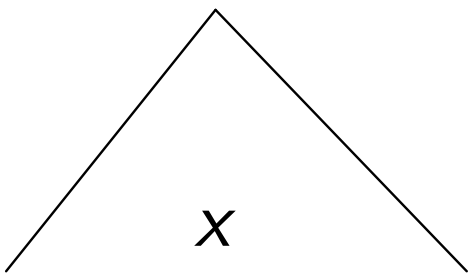

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$[\_, \_, \_]$

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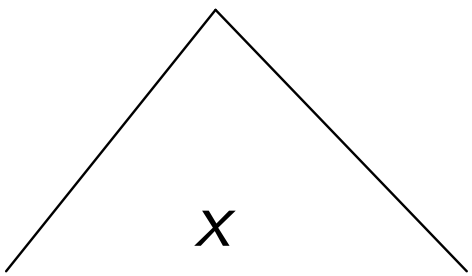

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Example

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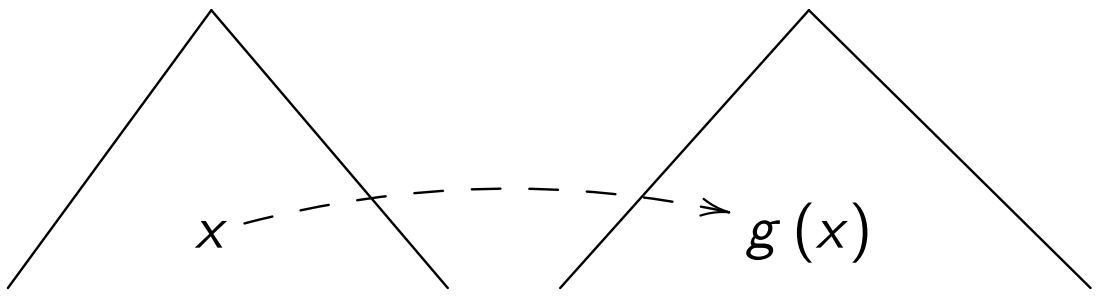
Ψ

$[x_1, x_2, x_3]$

3 Pillars: Boundedness, Naturality, **Functoriality** (BNF)

$$\begin{aligned} \mathsf{F} &: \mathsf{Set} \to \mathsf{Set} \\ \mathsf{map}_{\mathsf{F}} &: \prod_{A,B \in \mathsf{Set}} (A \to B) \to \mathsf{F}(A) \to \mathsf{F}(B) \end{aligned}$$

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & B \\ & & \\ \mathsf{F}(A) & \xrightarrow{\mathsf{map}_{\mathsf{F}}(g)} & \mathsf{F}(B) \\ \cup & & \cup \end{array}$$



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$$\begin{array}{ccc} \mathsf{F}\,(A) & \xrightarrow{\quad \mathsf{supp}_{\mathsf{F}} \quad} & \mathcal{P}\,(A) \\ \cup & & \cup \\ & & \{x, \ldots\} \\ & x & \end{array}$$

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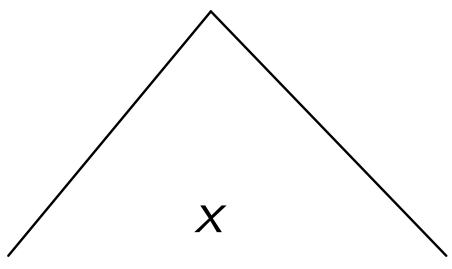
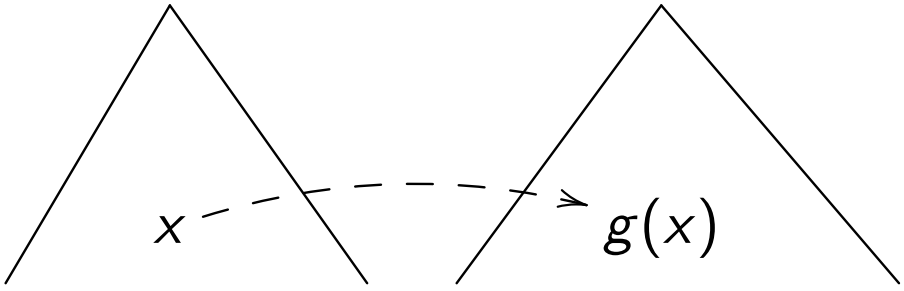
bd_{F} cardinal number

$$\begin{array}{ccc} \mathsf{F}\,(A) & \xrightarrow{\quad \mathsf{supp}_{\mathsf{F}} \quad} & \mathcal{P}\,(A) \\ \cup & & \cup \\ & & \mathsf{card}\,\{x, \ldots\} < \mathsf{bd}_{\mathsf{F}} \end{array}$$

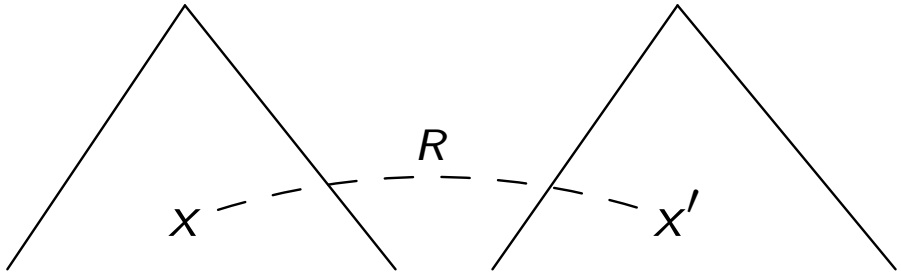


4th Pillar: Relator Structure

$$
\begin{array}{rrcl}
 & \mathsf{F} & : & \mathsf{Set} \to \mathsf{Set} \\
\text{Functor} & \mathsf{map}_{\mathsf{F}} & : & \prod_{A,B \in \mathsf{Set}} (A \to B) \to \mathsf{F}(A) \to \mathsf{F}(B) \\
\text{Relator} & \mathsf{rel}_{\mathsf{F}} & : & \prod_{A,B \in \mathsf{Set}} \mathcal{P}(A \times B) \to \mathcal{P}(\mathsf{F}(A) \times \mathsf{F}(B))
\end{array}
$$



$\mathsf{map}_{\mathsf{F}}(g)$
(slot-wise application of g)



$\mathsf{rel}_{\mathsf{F}}(R)$
(slot-wise lifting of R)

Example BNF: Lists

$$\mathsf{List} : \mathsf{Set} \to \mathsf{Set}$$

$\mathsf{map}_{\mathsf{List}}\ g\ [x_0, \ldots, x_{n-1}] = [g(x_0), \ldots, g(x_{n-1})]$

$\mathsf{supp}_{\mathsf{List}}\ [x_0, \ldots, x_{n-1}] = \{x_1, \ldots, x_{n-1}\}$

$\mathsf{bd}_{\mathsf{List}} = \aleph_0$

$([x_0, \ldots, x_{m-1}], [y_0, \ldots, y_{n-1}]) \in \mathsf{rel}_{\mathsf{List}}\ R$ iff
$m = n$ and $\forall i < m.\ (x_i, y_i) \in R$

Example BNF: Streams

$$\mathsf{Stream} : \mathsf{Set} \to \mathsf{Set}$$

$$\mathsf{map}_{\mathsf{Stream}}\ g\ [x_0, x_1, \ldots] = [g(x_0), g(x_1), \ldots]$$

$$\mathsf{supp}_{\mathsf{List}}\ [x_0, x_1, \ldots] = \{x_0, x_1, \ldots\}$$

$$\mathsf{bd}_{\mathsf{List}} = \aleph_1$$

$$([x_0, x_1, \ldots], [y_0, y_1, \ldots]) \in \mathsf{rel}_{\mathsf{List}}\ R \text{ iff}$$
$$\quad \forall i \in \mathbb{N}.\ (x_i, y_i) \in R$$

Other Examples of BNFs

Trees – finitely/infinitely branching, finite/infinite depth

Finite sets, countable sets, k-bounded sets

Multisets

Fuzzy sets

Probability distributions

...

Bounded Natural Functors (BNFs) in Isabelle

- BNFs
  - Include many useful container types
  - Closed under composition
  - Closed under least fixpoints (initial algebras)
  - Closed under greatest fixpoints (final coalgebra)

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Isabelle/HOL's (co)datatype package

"One of the greatest engineering projects since Stonehenge!"

Jasmin Blanchette

Dmitriy Traytel

Datatypes and Codatatypes Based on BNFs

Example: Interactive processes with final states in A, making queries in B and acting upon responses in C.

$$\begin{array}{l}\mathsf{datatype}\ \mathsf{Proc}(A, B, C) = \\ \quad \mathsf{Step}\ (A + B \times (C \to \mathsf{Proc}(A, B, C)))\end{array}$$

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Possibly nonterminating.

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Finitely nondeterministic?

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Possibly nonterminating.

Finitely nondeterministic? Plug in the finite powerset BNF.

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Finitely nondeterministic? Plug in the finite powerset BNF.

Countably nondeterministic? Plug in the countable powerset BNF.

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Countably nondeterministic? Plug in the countable powerset BNF.

Probabilistic? Plug in the probability distributions BNF.

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Possibly nonterminating.

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Probabilistic? Plug in the probability distributions BNF.

Nondeterminism plus probability? Plug in both BNFs.

Datatypes and Codatatypes Based on BNFs

For each defined (co)datatype, Isabelle provides

- operators: constructor, map, relator, support, etc.
- lemmas about these operators: injectiveness, functoriality, naturality ("free" theorems), etc.
- (co)recursion definition principles
- (co)induction proof principles

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... typically not provided automatically by proof assistants

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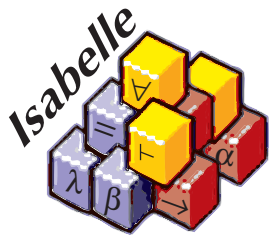


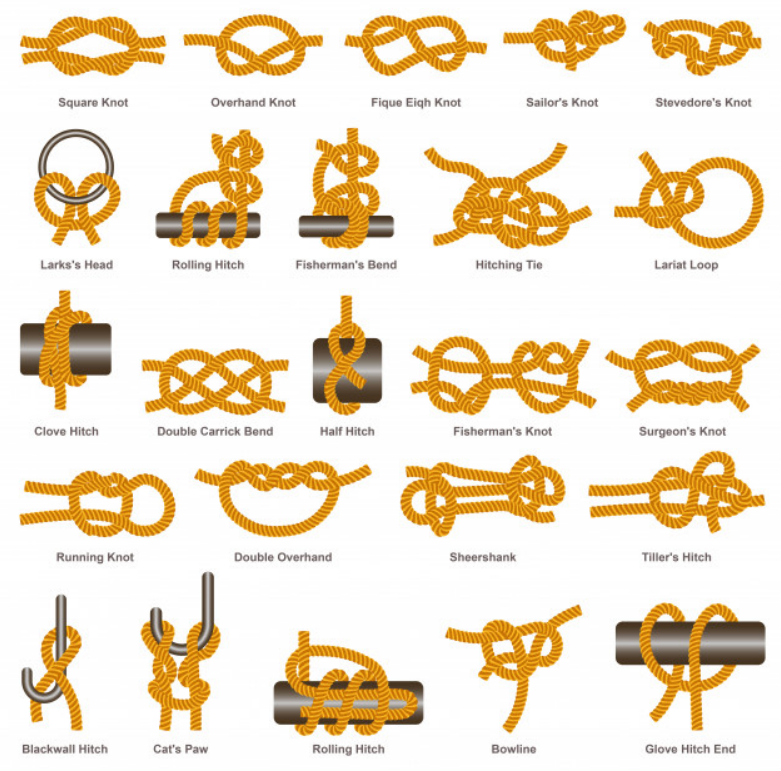
BNFs lurking in the background
without the users knowing of them

A Foundation for Binders

What is a Binder?

Several sophisticated syntactic formats:

- CαML [Pottier 2006], Ott [Sewell et al. 2010], Unbound [Weirich et al. 2011], Isabelle Nominal2 [Urban and Kaliszyk 2012], Needle & Knot [Keuchel et al. 2016]



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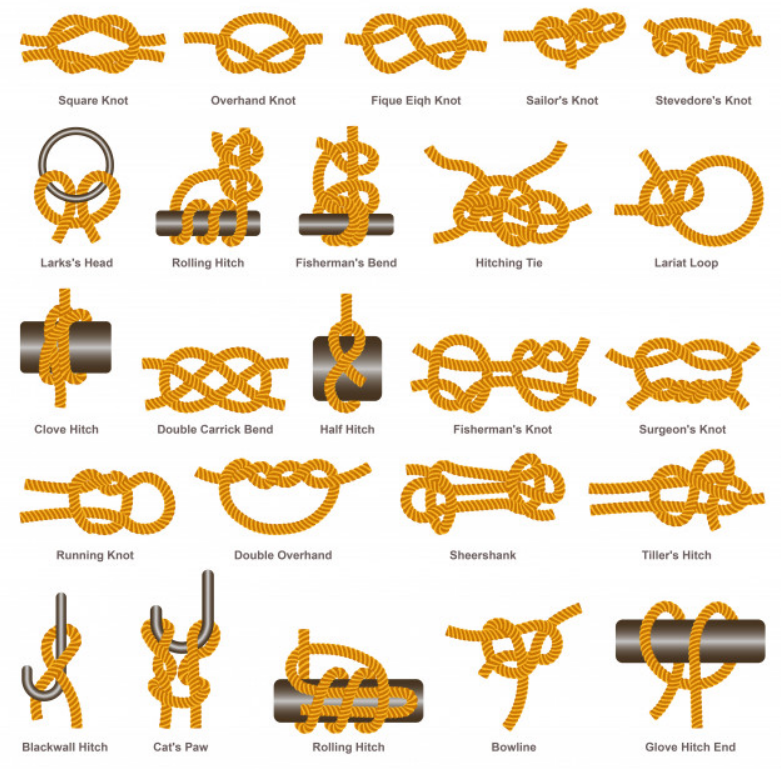

Capture essence without committing to a particular syntax?

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Binder = Mechanism for combining any variables with any terms.

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let $v = t_1$ in t_2

let rec $v_1 = t_1$ and $\ldots$ and $v_k = t_k$ in t

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Proposal: Binder = Operator on sets $F : \mathrm{Set}^m \times \mathrm{Set}^n \to \mathrm{Set}$
plus binding dispatcher relation $\theta \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$.

Think: $F\,(V_1, \ldots, V_m, T_1, \ldots, T_n)$ combines variables $v_i \in V_i$ and terms $t_j \in T_j$ such that $v_i \in V_i$ binds in $t_j \in T_j$ if $(i, j) \in \theta$.

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$m = n = 1$
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$\mathrm{let}\ v = t_1\ \mathrm{in}\ t_2$	$m = 1,\ n = 2$ $\mathrm{F}\,(V, T_1, T_2) = V \times T_1 \times T_2$ $\theta = \{(1, 2)\}$

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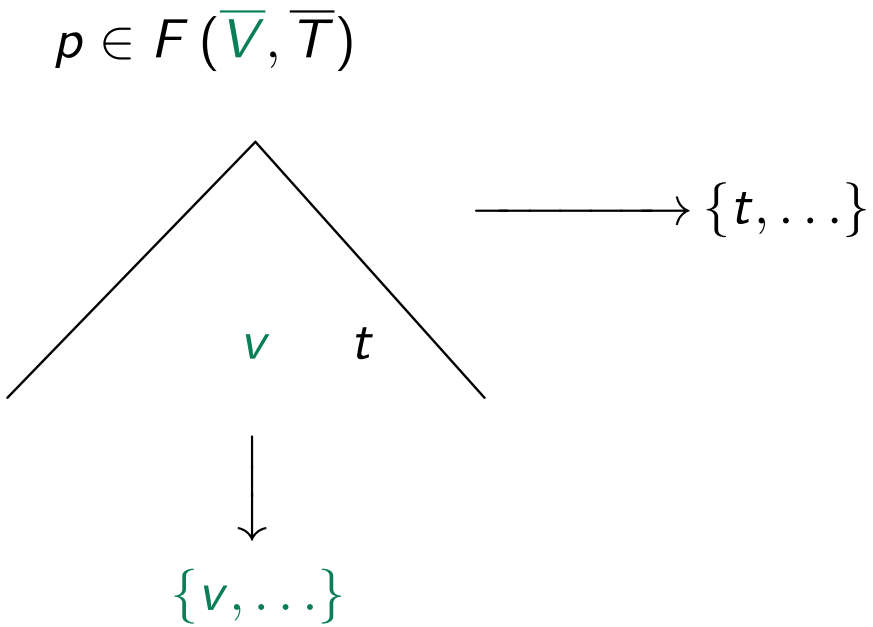

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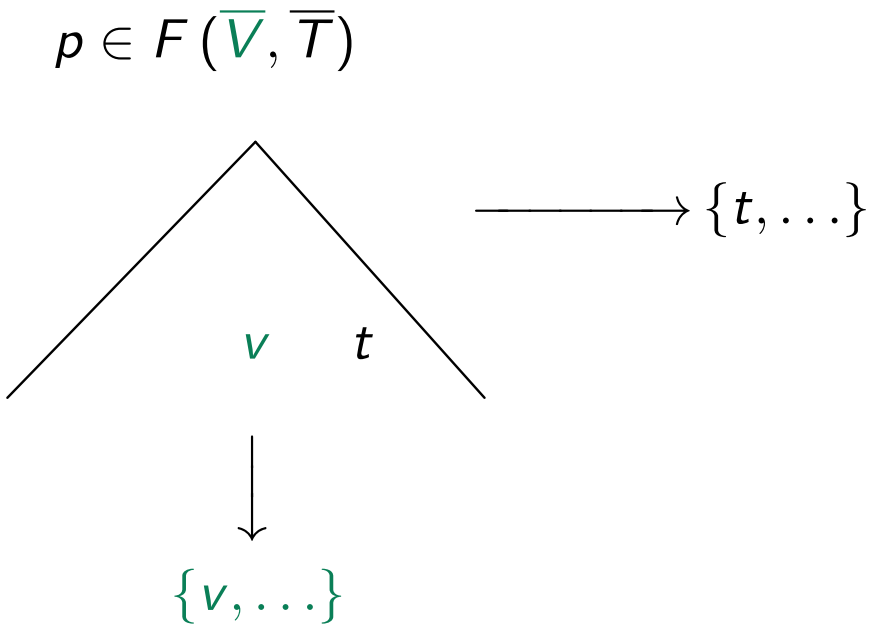

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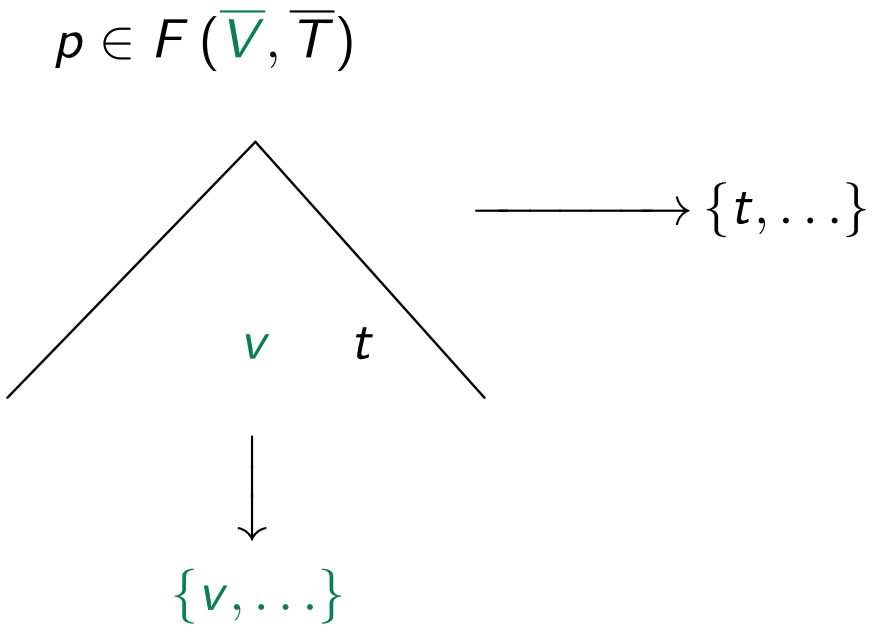

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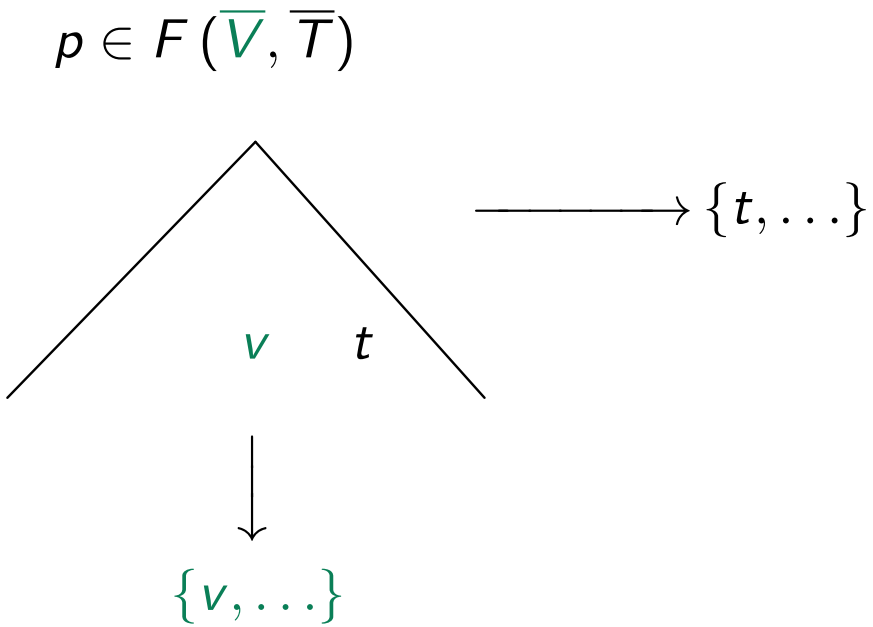

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$\mathsf{let\ rec}\ (v = t)^*\ \mathsf{in}\ t$	$\mathsf{F}\,(V, T) =$ $\mathsf{List}\,(V \times T) \times T$ $\theta = \{(1,1)\}$	$\mathsf{let\ rec}\ v = t_1\ \mathsf{and}\ v = t_2\ \mathsf{in}\ t$ $[(v, t_1), (v, t_2)] \in \mathsf{List}\,(V \times T)$

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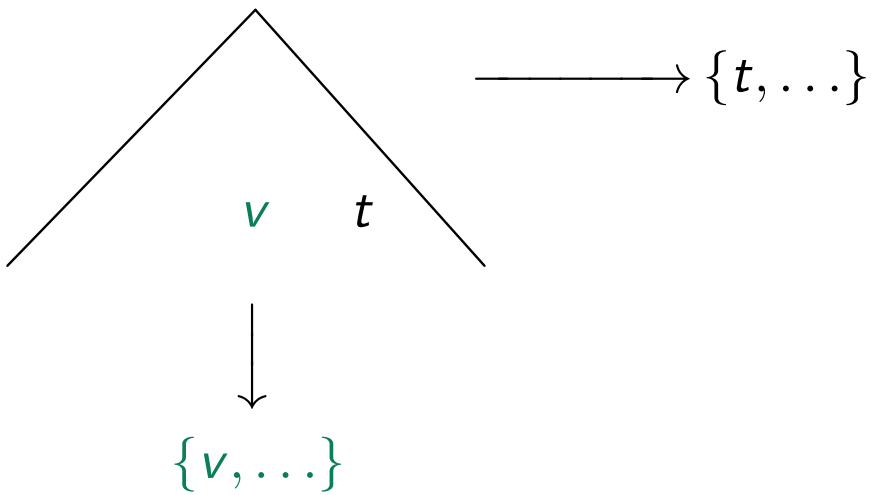

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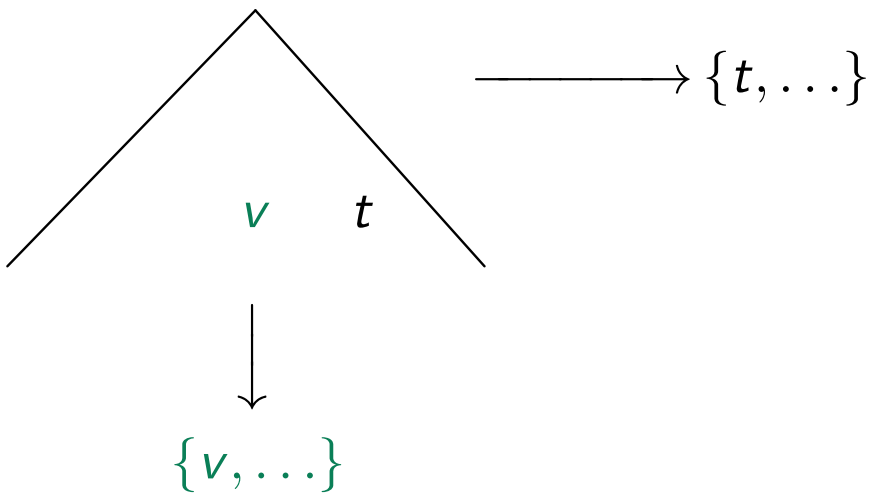

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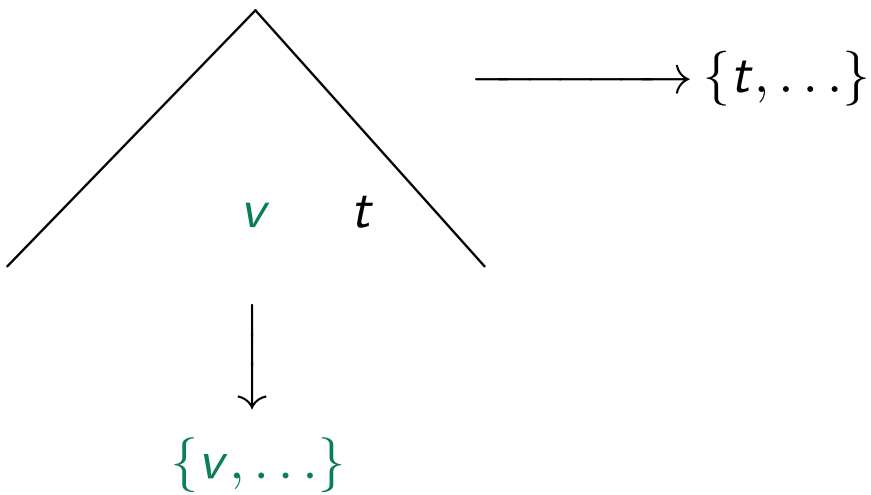

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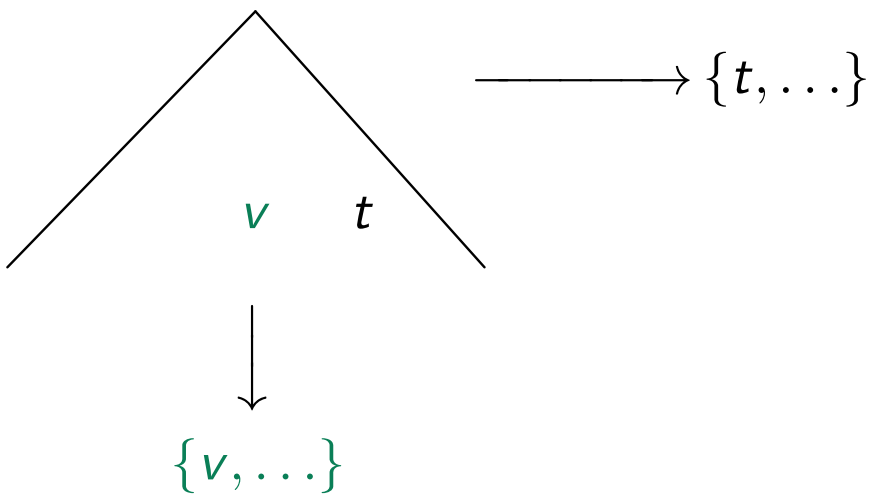

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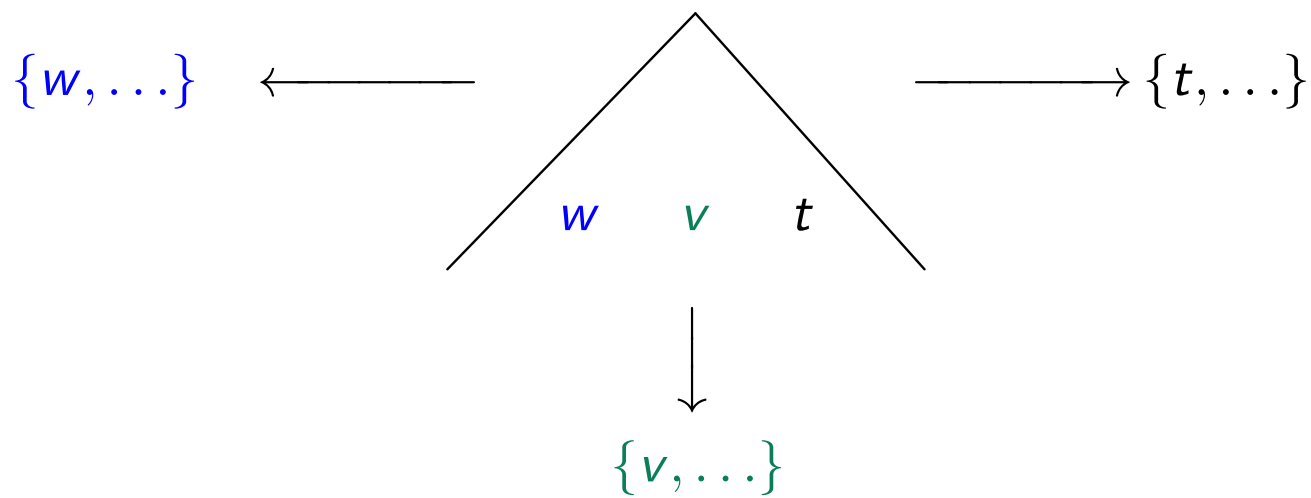

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$$\mathsf{List}\,(A)^{@A} = \{xs \in \mathsf{List}\,(A) \mid \forall i, j.\ i \neq j \longrightarrow xs_i \neq xs_j\}$$

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How about: $p \in F\,(A)$ linear
... if $\forall q.\ \mathsf{shape}\,(q) = \mathsf{shape}\,(p) \longrightarrow |\mathsf{supp}_{\mathsf{F}}\,(q)| \leq |\mathsf{supp}_{\mathsf{F}}\,(p)|$

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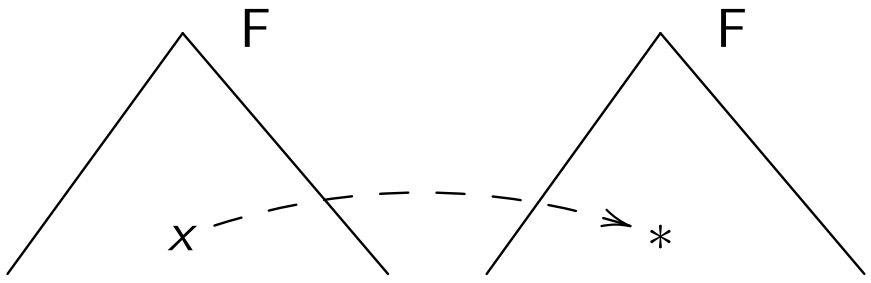
$$\begin{aligned} !_A &: A \to \mathsf{Unit} = \{*\} \\ \mathsf{shape} = \mathsf{map}_{\mathsf{F}}\, !_A &: \mathsf{F}\,(A) \to \mathsf{F}\,(\mathsf{Unit}) \end{aligned}$$



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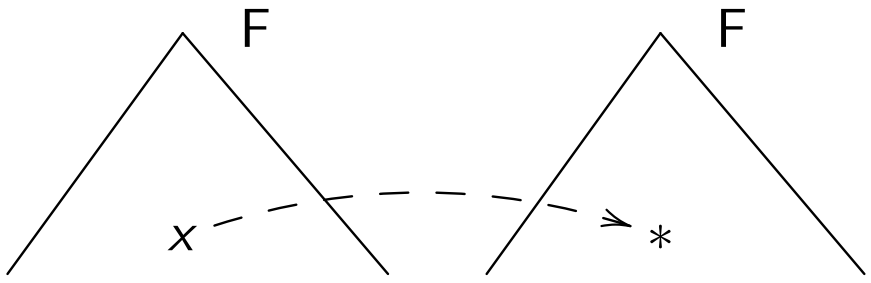

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Works for finitary functors.

Fails in general: For $\mathsf{F} = \mathsf{Stream}$, $[0, 0, 1, 2, 3, \ldots] \in \mathsf{F}\,(\mathbb{N})$ linear.

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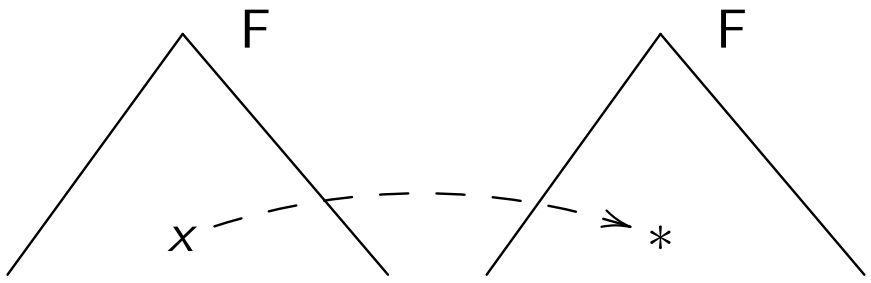

Better: $p \in F\,(A)$ linear
... if $\forall q.\ \mathsf{shape}\,(q) = \mathsf{shape}\,(p) \longrightarrow \exists f : A \to A.\ \mathsf{map}_{\mathsf{F}}\,(f)\,(p) = q$

Works in general.

Gives us back a sub-functor, $\mathsf{F}^{@}$, of F's restriction to bijections.

Structure of Binders (Summary)

Proposal: Binder = Operator on sets $F : \mathrm{Set}^p \times \mathrm{Set}^m \times \mathrm{Set}^n \to \mathrm{Set}$ that is a Map-Restricted Bounded Natural Functor (MRBNF):

- w.r.t. arbitrary functions on the p free-variable arguments
- w.r.t. injections on the m binding-variable arguments
- w.r.t. arbitrary functions on the n "term" arguments

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Type-variable α, term-variables x, labels l

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\text{Patterns} & p & ::= & x : \sigma \mid \{l_i = p_i \,^{i \in 1 \ldots n}\} \\
\text{Terms} & t & ::= & x \mid \Lambda\alpha.\, t \mid \text{let } p = t_1 \text{ in } t_2
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In terms, term-variables coming from patterns and type-variables near Λ's are binding.

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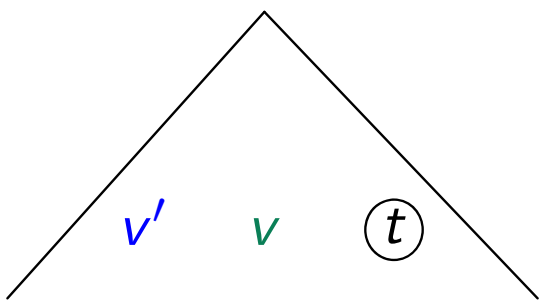
From Binders to Terms with Bindings

Constructing Terms from Binders

$$F : \mathsf{Set}^{p} \times \mathsf{Set}^{m} \times \mathsf{Set}^{n} \to \mathsf{Set}$$

Assume $p = m$.

$$\mathsf{T}\,(\overline{V}) \;=\; \mu\overline{A}.\; \mathsf{F}(\overline{V}, \overline{V}, \overline{A})$$



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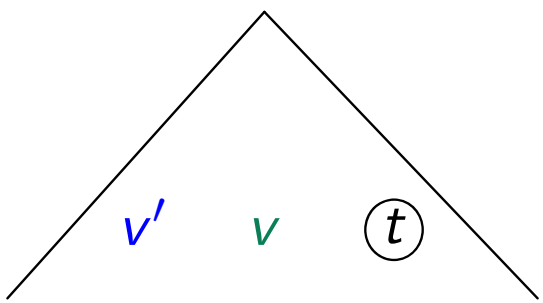

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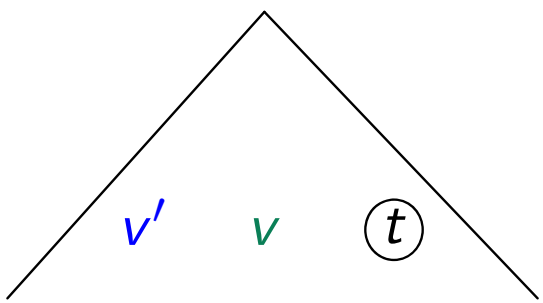
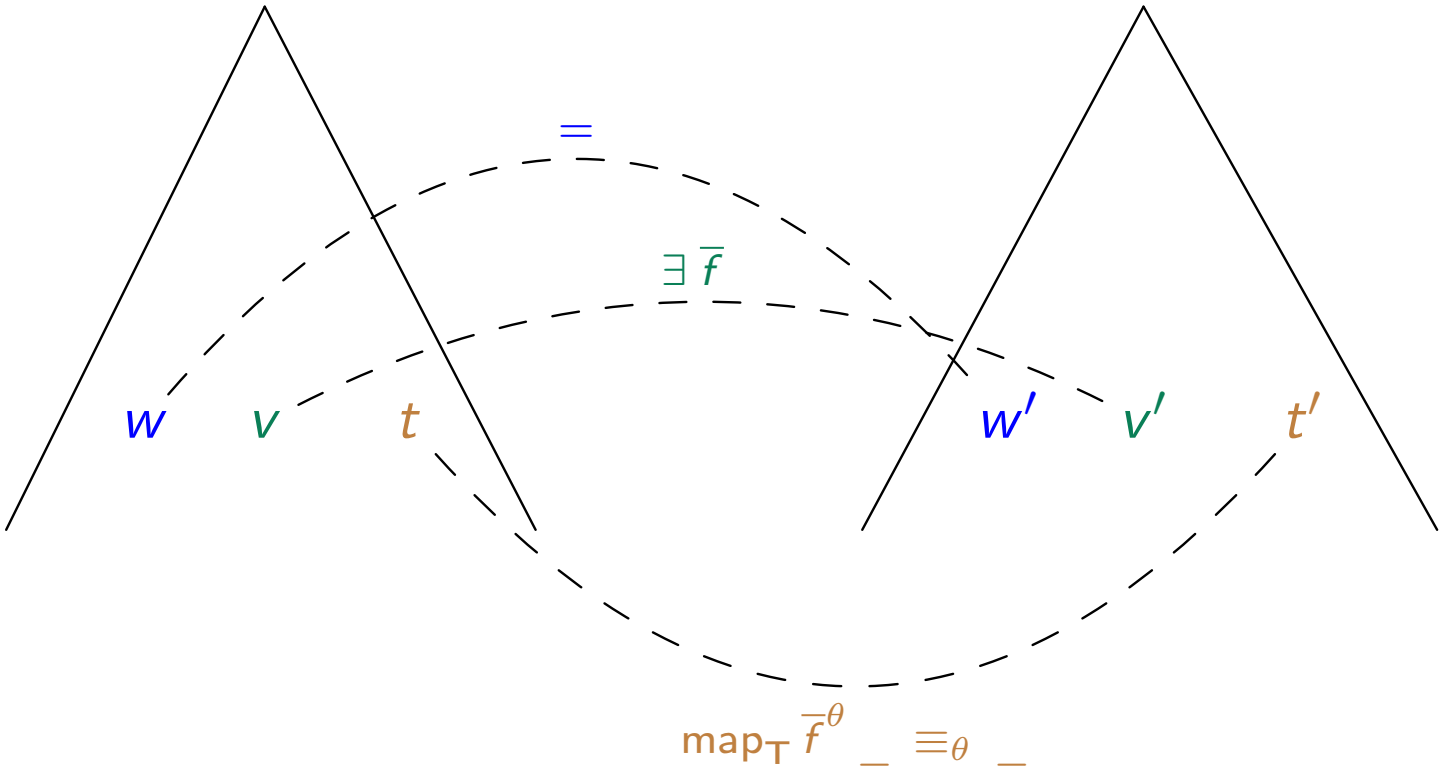


Alpha-quotiented terms: $\overline{\mathbf{T}\,(\overline{V})} \;=\; \overline{\mathsf{T}\,(\overline{V})\,/{\equiv_{\theta}}}$

Inductive Definition of Alpha-Equivalence



Equality on the top free variables

Possible bijective renamings of top binding variables

Recursive call factoring in the renamings

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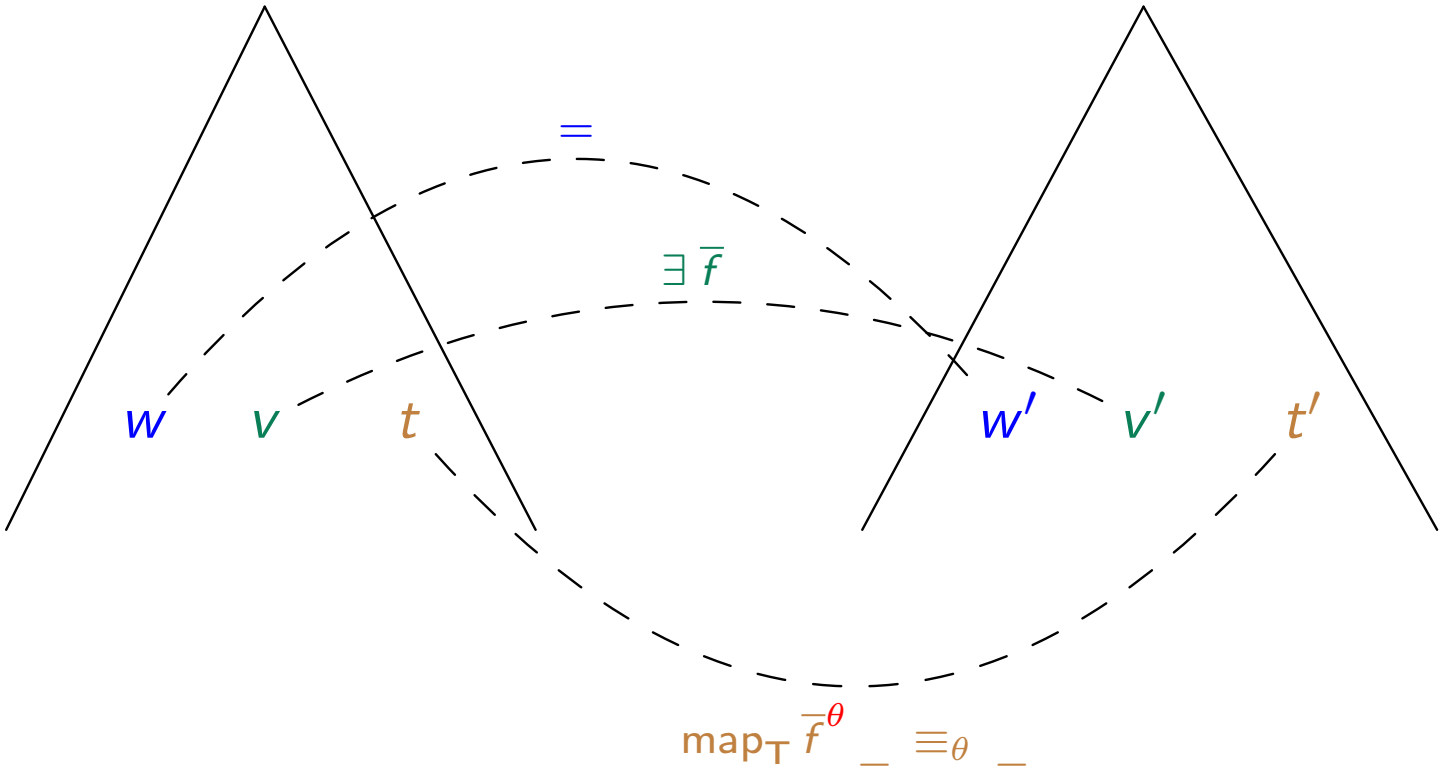

Renaming via f_i of v_i in t_j only if $(i, j) \in \theta$

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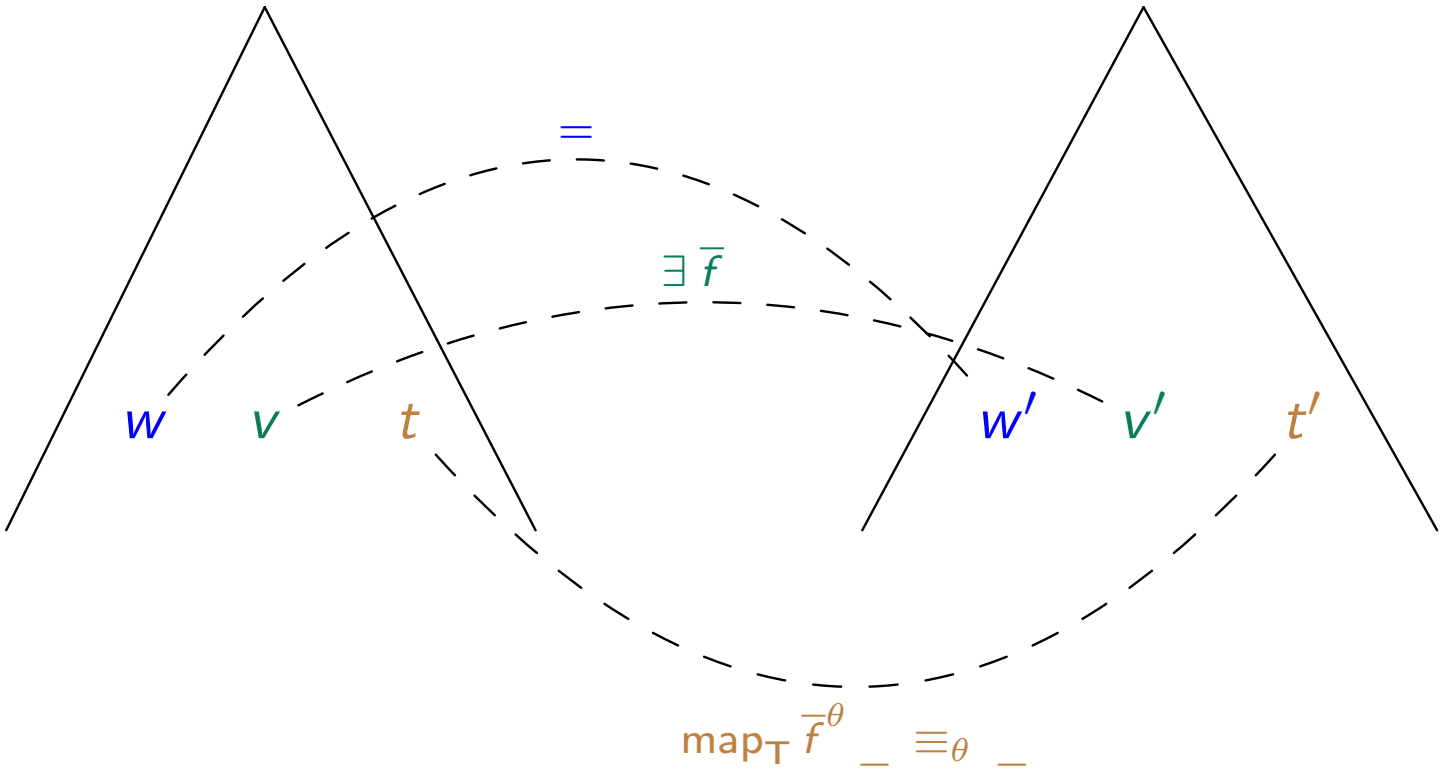
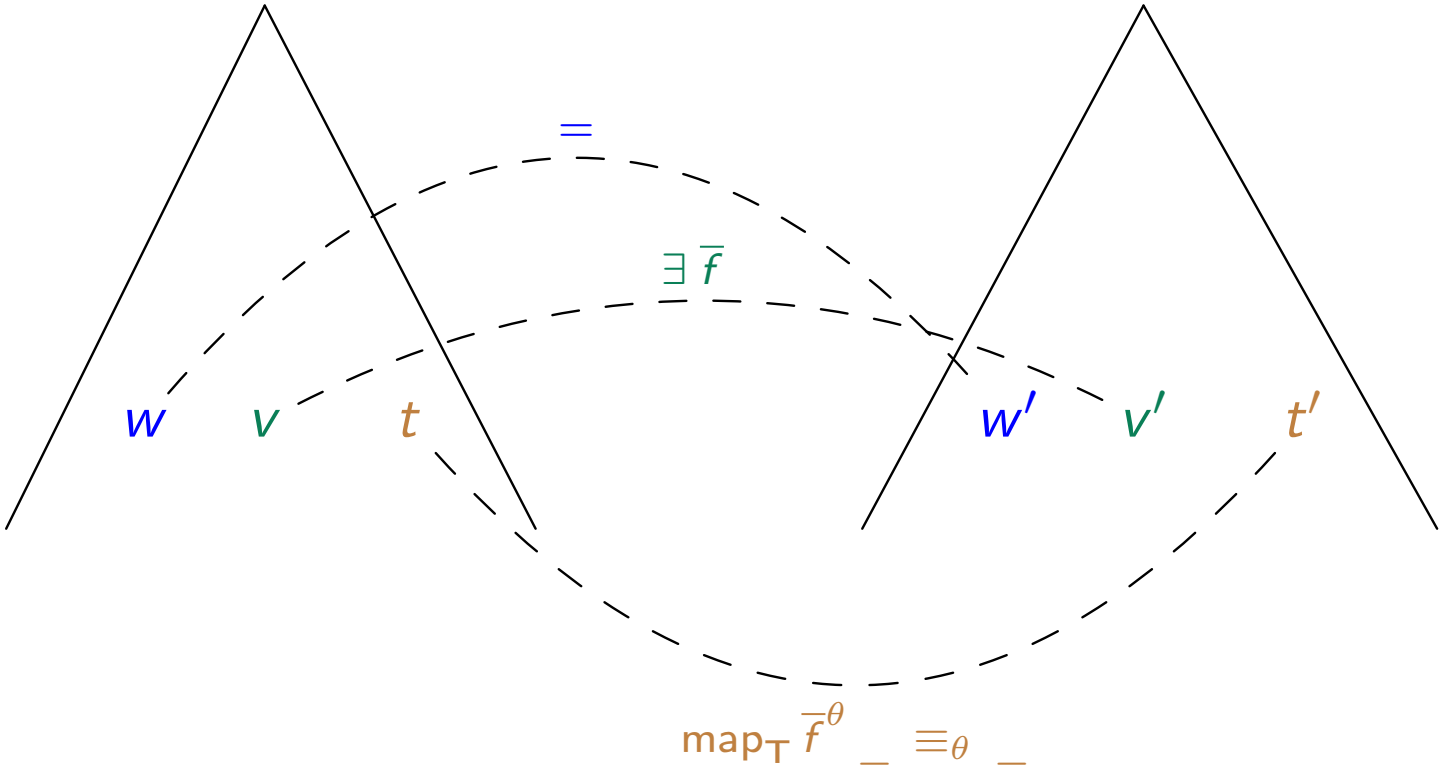


F being a relator is crucial:

- Equality on the top free variables
- Possible bijective renamings of top binding variables
- Recursive call factoring in the renamings

Inductive Definition of Alpha-Equivalence



F being a relator is crucial:

$$\frac{(\mathsf{unf}(t), \mathsf{unf}(t')) \in \mathsf{rel}_{\mathsf{F}}\ (=)\ \{(\overline{v}, \overline{f\ v}) \mid \ldots\}\ \{(\overline{t}, \overline{t'}) \mid \mathsf{map}_{\mathsf{T}}\ \overline{f}^{\theta}\ \overline{t} \equiv_{\Theta} \overline{t'}\}}{t \equiv_{\theta} t'}$$

Abstract Characterization of Alpha-Quotinented Terms?

$$
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\mathsf{T}\,(\overline{V}) & = & \mu A.\ \mathsf{F}(\overline{V}, \overline{V}, A) & \text{OK} \\
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Theorem: $(\mathbf{T}, \overline{\mathsf{FVars}}, \mathsf{map}_{\mathbf{T}}, \mathsf{ctor})$ is the initial object in a category of models $\mathcal{U} = (\mathsf{U}, \overline{\mathsf{UFVars}}, \mathsf{Umap}, \mathsf{Uctor})$ satisfying:

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Recursor generalizing the state-of-the-art nominal recursors (Norrish 2004, Pitts 2006, Urban and Berghofer 2006, GP 2017)

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Example – HOAS encoding

$\# : \mathsf{Term}_\lambda \to \mathsf{Term}_\lambda(\mathsf{app}, \mathsf{lam})$

(1) $x^\# = x$

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Why is this a correct recursive definition?

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End product: $\exists!$ # satisfying (1-4), and (5) or (5') or both.

Parenthesis: How the Recursors Work

Example: Interpretation of FOL formulas

sem : Fmla $\to$ (Var $\to$ M) $\to$ Bool

sem $(\forall x.\ \varphi)$ (ξ) defined as

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"That was not too hard: I have my (α-preserving) semantic interpretation defined, its dependence on free vars proved, and its substitution lemma proved, all in one go. Now I can move on and do interesting things."

Infinitary Terms with Bindings?

Occasionally useful

- Infinitely branching process sums in Milner's CSS: $\sum_{i \in I} P_i$
- Infinitary logics
- Böhm trees: λ-terms with possibly infinite depth
- Fully abstract π-calculus trees (via "unfolding" process terms)

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Already in the scope of what I've shown
Covered by binding-aware greatest fixpoints

Binding-Aware Greatest Fixpoints?

Recall the Binding-Aware Least Fixpoints

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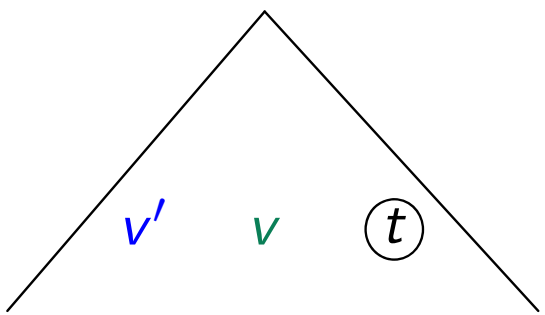

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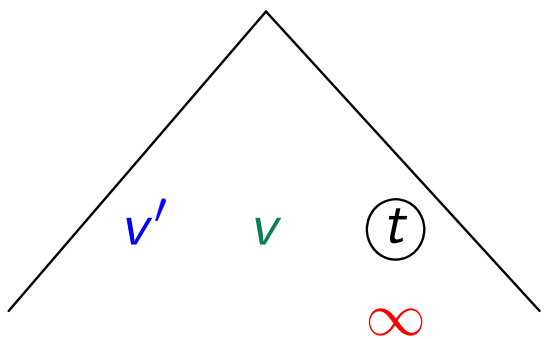

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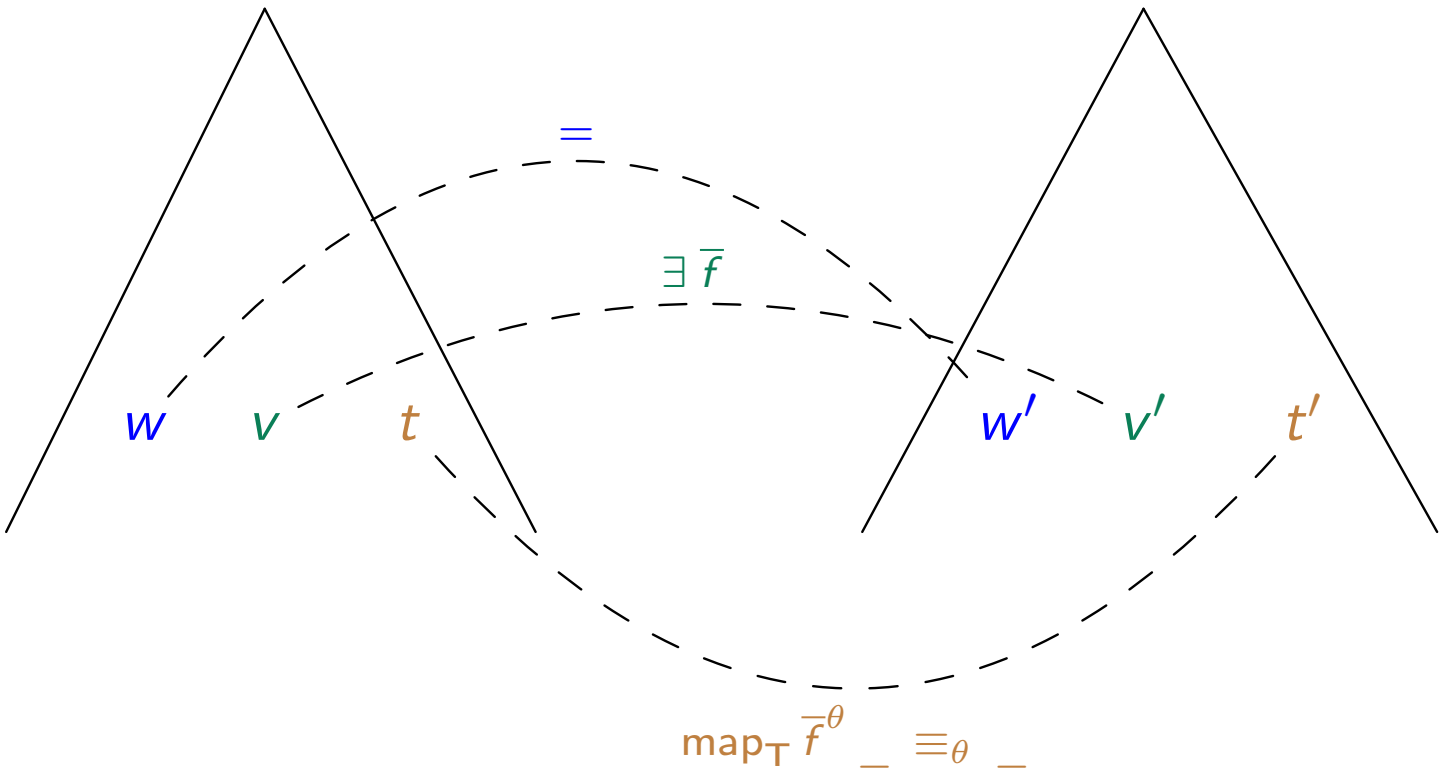


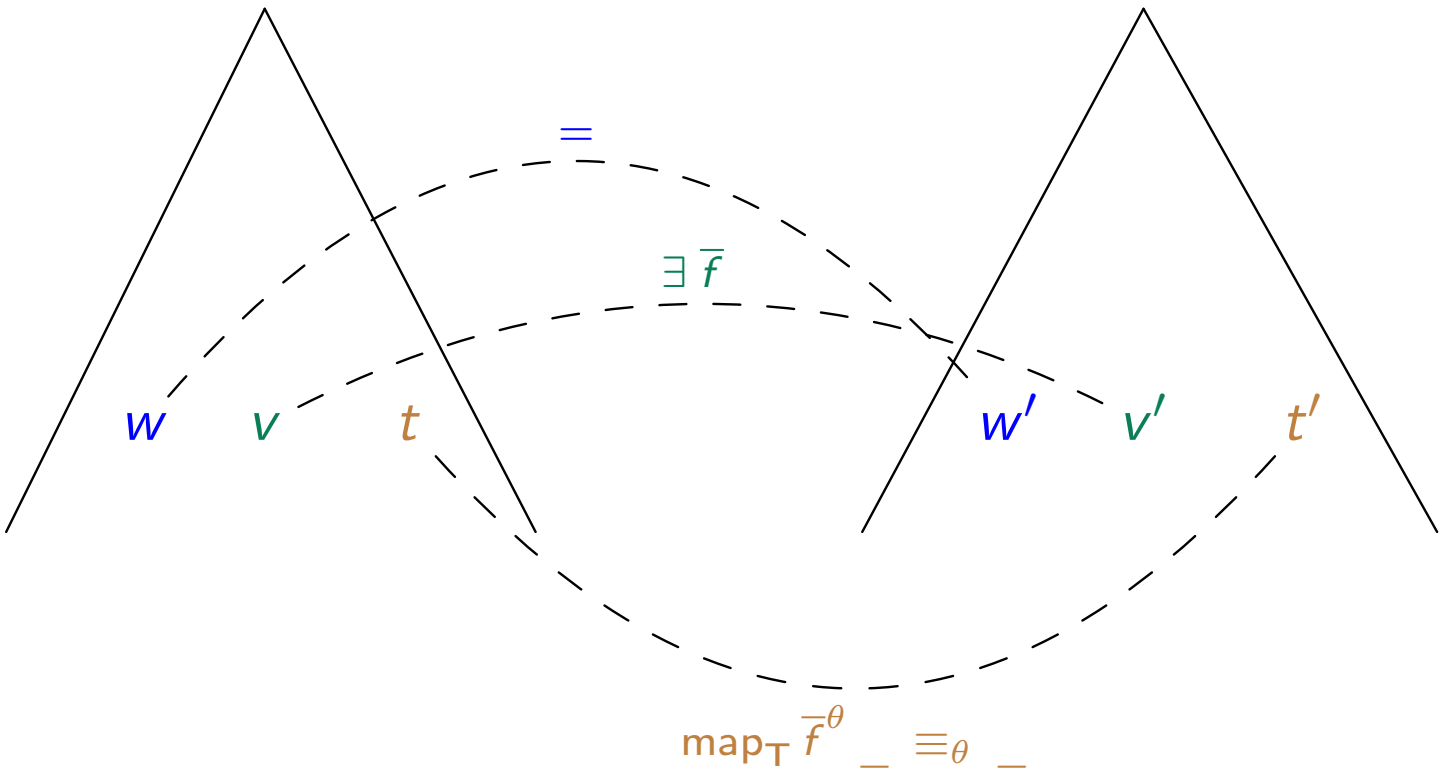
Equality on the top free variables

Possible bijective renamings of top binding variables

Recursive call factoring in the renamings

Coinductive Definition of Alpha-Equivalence

Same characteristic clause, same intuition,
... but GFP instead of LFP in Knaster-Tarski



Equality on the top free variables
Possible bijective renamings of top binding variables
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MRBNFs = BNFs with Binding Awareness

BNFs

- Include many useful container types
- Closed under composition
- Closed under least fixpoints
- Closed under greatest fixpoints

Compositional (co)datatypes
Implemented in Isabelle: user-friendly, hides category theory

MRBNFs = BNFs with Binding Awareness

MRBNFs

- Include many useful container types
- Closed under composition
- Closed under least fixpoints
- Closed under greatest fixpoints
- Closed under binding-aware least fixpoints
- Closed under binding-aware greatest fixpoints
- Closed under linearization

Compositional binding-aware (co)datatypes

Isabelle: worked out category theory, not yet user-friendly

References

Expressive datatypes and codatatypes:

- Foundational, Compositional (Co)datatypes for HOL. LICS'12
- Truly Modular (Co)datatypes for Isabelle/HOL. ITP'14
- Foundational nonuniform (Co)datatypes for HOL. LICS'17
- Relational Parametricity and Quotient Preservation for Modular (Co)datatypes. ITP'18
- Quotients of Bounded Natural Functors. IJCAR'20
  (TBP tomorrow)

Ensuring non-emptiness of types:

- Witnessing (Co)datatypes. ESOP'15

Expressive function definition mechanisms:

- Foundational extensible corecursion. ICFP'15
- Corecursion in Foundational Proof Assistants. ESOP'17

Overview of entire line of work

- Foundational (Co)datatypes and (Co)recursion for HOL. FroCoS'17

Supernominal (MRBNF extension of BNF)

- Bindings as Bounded Natural Functors. POPL'19

Related Work

Nominal: Logic, Techniques, Datatypes

Murdoch J. Gabbay'

Michael Norrish

Andrew Pitts

Christian Urban

The Super in Supernominal

	Nominal Datatypes	MRBNFs (Binders as Functors)
BA-Induction	●	●
BA-Recursion	◒	●
Infinite Branching		●
Coinductive Types	◔	●
BA-Coinduction		●
BA-Corecursion	◔	●
Complex Binders	◒	●
Modularity	◔	●

BA = binding-aware

The Super in Supernominal

Also, Transnominal: Beyond Finite Support

	Nominal Datatypes	MRBNFs (Binders as Functors)
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BA = binding-aware

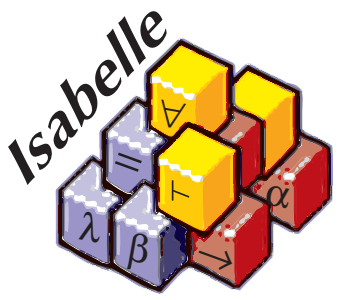
Binding-Aware Induction and Recursion

Supernominal Recursors

Urban'08 Norrish'04 Gheri&Popescu'17
Pitts'05 Gordon&Melham'96 Popescu&Gunter'11

Prior work on nominal corecursion: Kurz et al. 2013
Supernominal lifts their restriction to finite support

Syntax with Bindings in Isabelle



Isabelle Nominal2 [Urban and Kaliszyk 2012]

- Good user support
- Complex binders via syntactic format

Supernominal (not yet fully implemented)

- Will boost expressiveness and compositionality
- Will stay backwards-compatiblish with Nominal/Nominal2

1999: The Year the Earth Stood Still

Much category theory on De Bruijn style, starting with

- Fiore et al. (LICS'99)
- Hofmann (LICS'99)
- Bird and Paterson (J. Func. Prog. '99)
- Altenkirch and Reus (CSL'99)

The same year: Nominal Logic – Gabbay and Pitts (LICS'99)

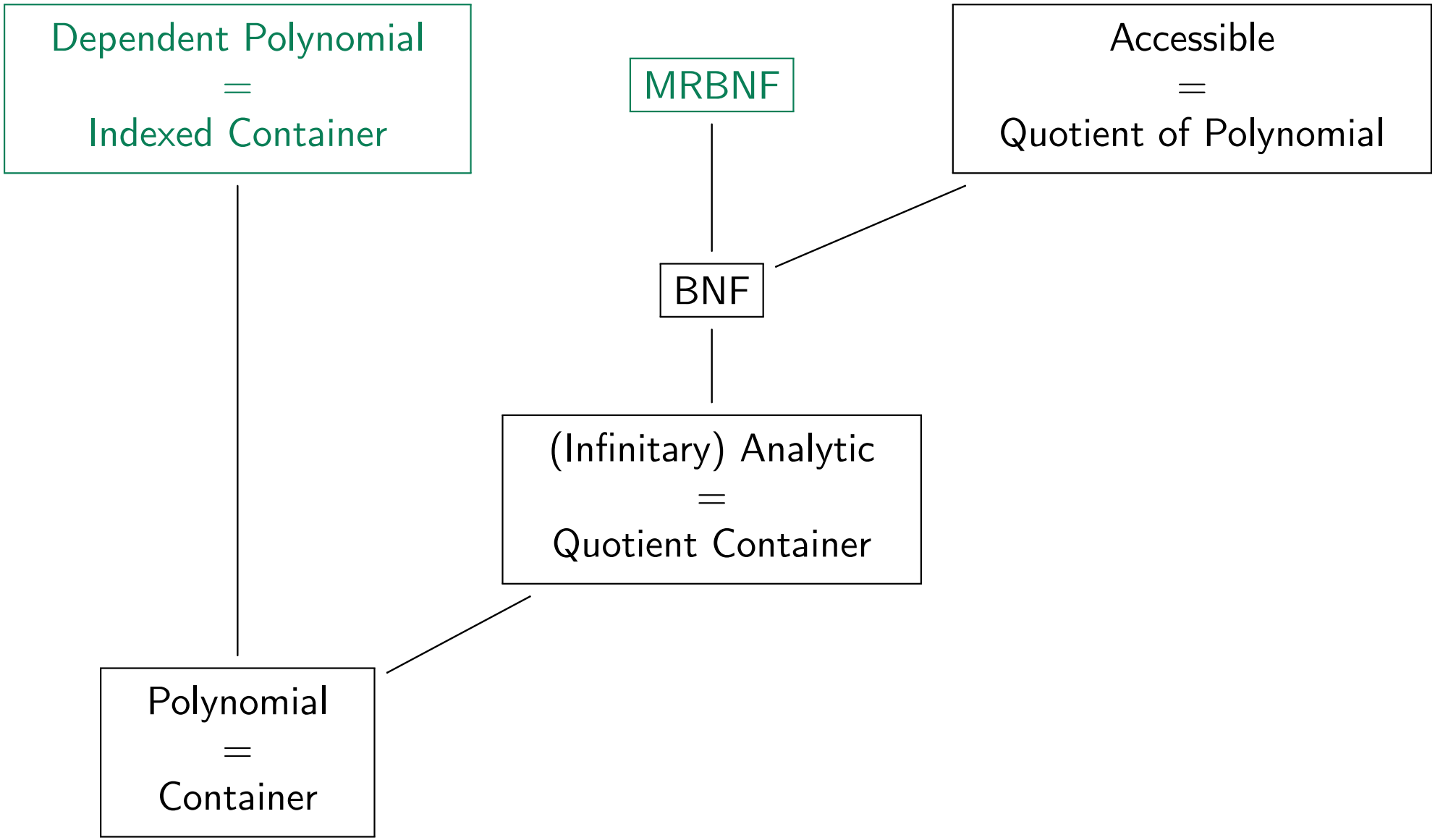
Precursors of BNFs

BNFs = subclass of k-accessible functors

Container types [Hoogendijk and de Moor 2000]

Containers [Abbott et al. 2005]

Relevant Classes of Functors



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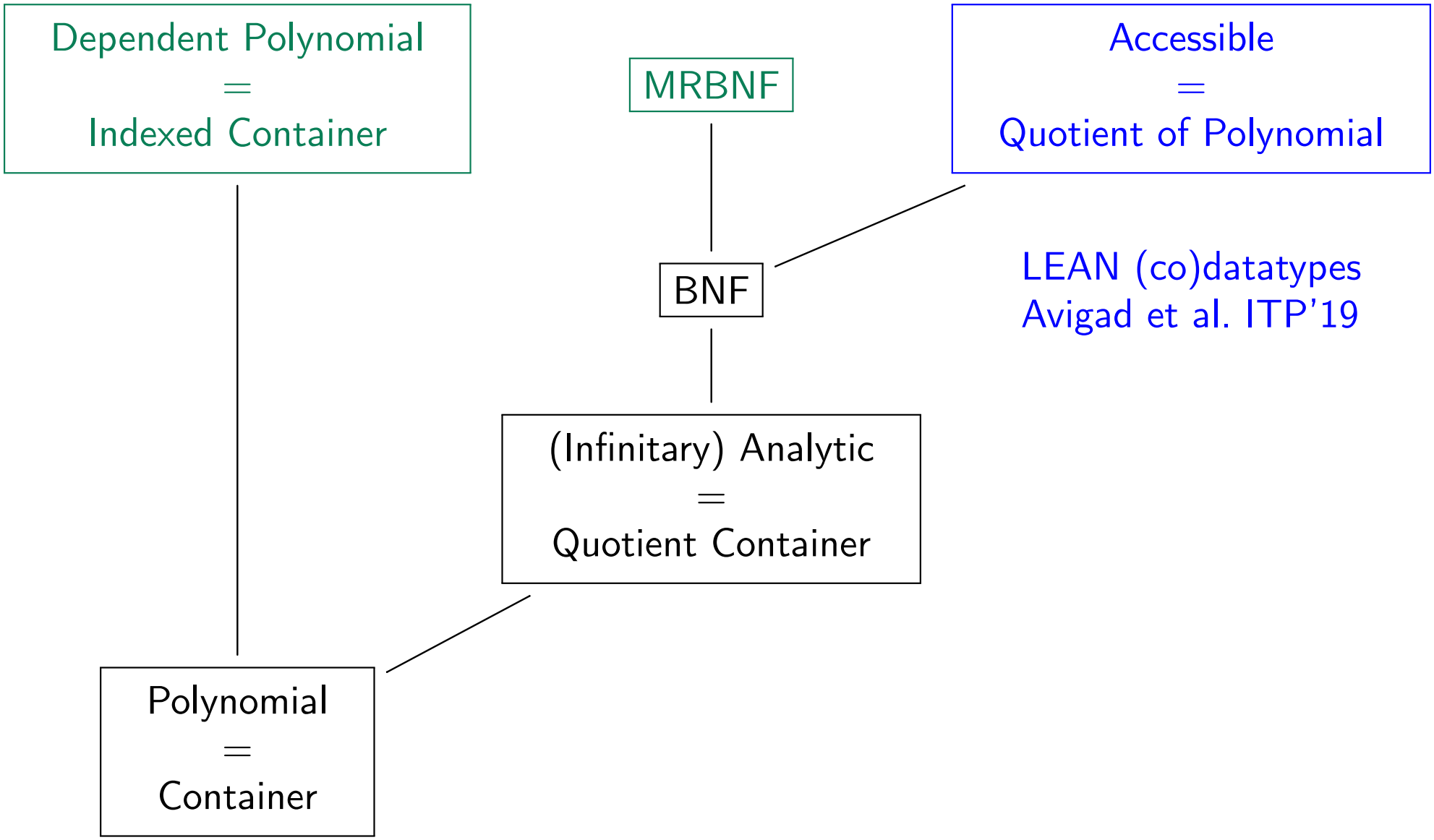

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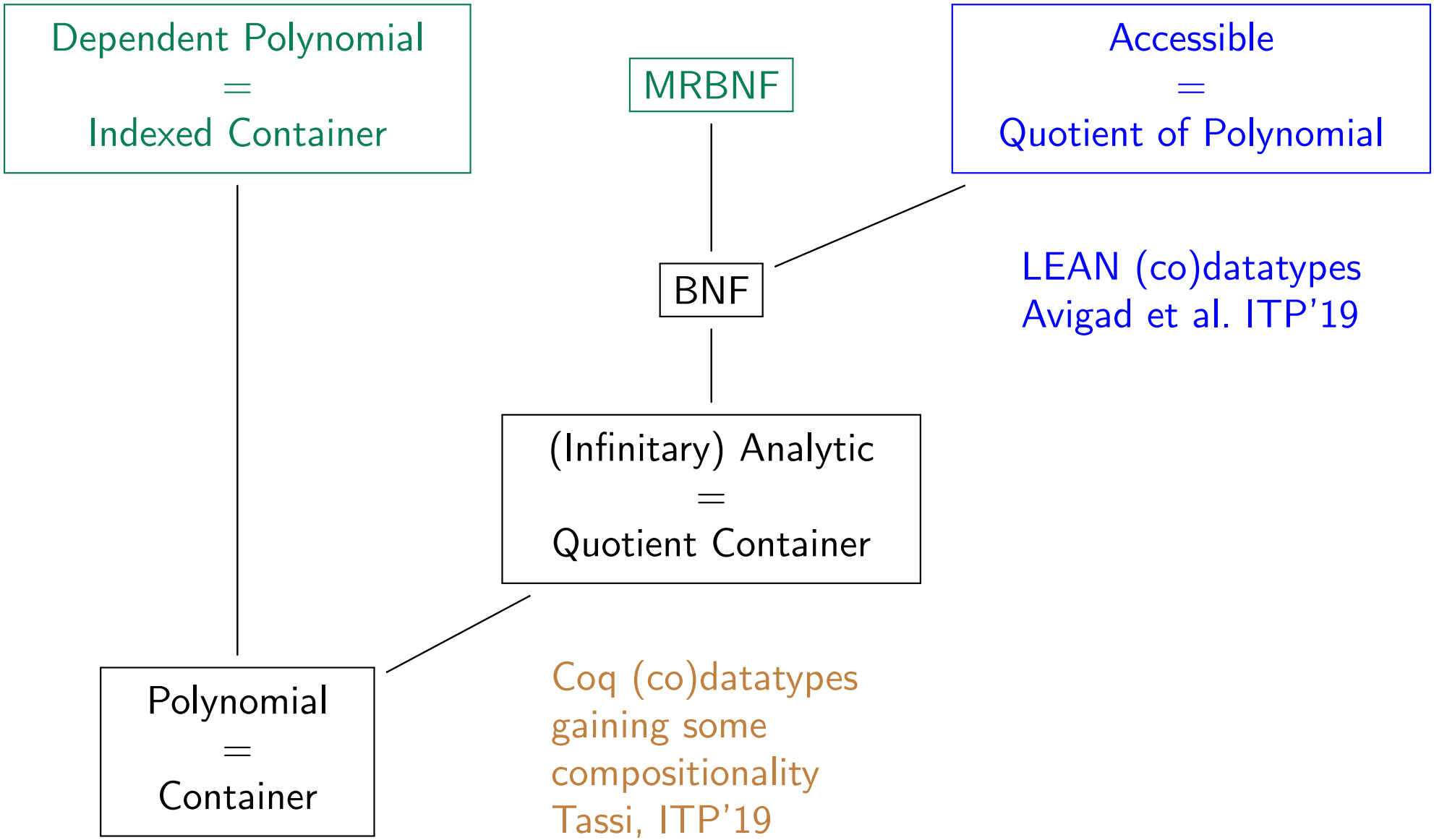

Bindings Are Functors

Bindings Are Functors

Andrei Popescu
University of Sheffield, UK
LFMTP
June 30, 2020

Joint work with

Jasmin Blanchette, Lorenzo Gheri, Dmitriy Traytel, Isabelle/HOL