A Semantic Approach to Interpolation

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Abstract

Craig interpolation is investigated for various types of formulae. By shifting the focus from syntactic to *semantic interpolation*, we generate, prove and classify a series of interpolation results for first-order logic. A few of these results non-trivially generalize known interpolation results; all the others are new. We also discuss some applications of our results to the theory of institutions and of algebraic specifications, and a Craig-Robinson version of these results.

Key words: Craig Interpolation, Birkhoff-style axiomatizability, Craig-Robinson Interpolation, First-Order Sub-Logics, Algebraic Specifications, Institutions.

1 Introduction

Craig interpolation is a landmark result in first-order logic [8]. In its original formulation, it says that given sentences Γ_1 and Γ_2 such that $\Gamma_1 \models \Gamma_2$, [1] there is some sentence Γ whose non-logical symbols occur in *both* Γ_1 and Γ_2, called an

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[1] In this paper we adopt the usual convention to let $\models$ denote *semantic* deduction (just like the satisfaction relation) and $\vdash$ *syntactic* deduction. Thanks to complete-

interpolant, such that $\Gamma_1 \models \Gamma$ and $\Gamma \models \Gamma_2$. This well-known result can also be rephrased as follows: given first-order signatures Σ_1 and Σ_2, a Σ_1-sentence Γ_1 and a Σ_2-sentence Γ_2 such that $\Gamma_1 \models_{\Sigma_1 \cup \Sigma_2} \Gamma_2$, there is some $(\Sigma_1 \cap \Sigma_2)$-sentence Γ such that $\Gamma_1 \models_{\Sigma_1} \Gamma$ and $\Gamma \models_{\Sigma_2} \Gamma_2$. The conclusion of studying interpolation in various *extensions* of first-order logic was that "interpolation is indeed [a] rare [property in logical systems]" ([2], page 68). We show in this paper that the situation is totally different when one looks in the opposite direction, at *restrictions* of first-order logic: there is a plethora of interpolation results.

There are simple sub-logics of first-order logic, such as equational logic, where the interpolation result does *not* hold for sentences, but it holds for *sets of sentences* [35]. For this reason, as well as for reasons coming from theoretical software engineering, in particular from specification theory and modularization [3,15,16,10], it is quite common today to state interpolation more loosely, in terms of sets of sentences Γ_1, Γ_2, and Γ. This is also the approach that we follow in this paper.

We call our approach to interpolation "semantic" because we shift the problem of finding syntactic interpolants Γ to a problem of finding appropriate *classes of models*, which we call *semantic interpolants*. We present a precise characterization for *all* the semantic interpolants of a given instance $\Gamma_1 \models_{\Sigma_1 \cup \Sigma_2} \Gamma_2$, as well as a general theorem ensuring the existence of semantic interpolants closed under generic closure operators. Not all semantic interpolants correspond to sets of sentences. However, when semantic interpolants are closed under certain operators, they become *axiomatizable*, thus corresponding to some sets of sentences. Following the fruitful idea from [35] of proving, for equational logic, Craig interpolation from the Birkhoff axiomatizability theorem, a similar semantic approach was investigated in [32], but it was only applied there to obtain Craig interpolation results for categorical generalizations of equational logics. A similar idea is employed in [10], where interpolation results are presented in an institutional [21] setting. While the institution-independent interpolation results in [10] can potentially be applied to various particular logics, their instances still refer to just one type of sentence: the one that the particular logic comes with.

The conceptual novelty of our semantic approach to interpolation in this paper is to keep the restrictions on Γ_1, Γ_2, and Γ, or more precisely the ones on their corresponding classes of models, *independent*. This way, surprising and interesting results can be obtained with respect to the three types of sentences involved. By considering several combinations of closure operators allowed by our parametric semantic interpolation theorem, we provide many interpolation results; some of them generalize known results, but most of them are new. For

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ness, semantic and syntactic deductions coincide for first-order logic and for its discussed sub-logics.

example, we show that if the sentences in Γ_1 are first-order while the ones in Γ_2 are universally quantified Horn clauses (UHC's), then those in the interpolant Γ can be chosen to be UHC's, too. Surprisingly, sometimes the interpolant is strictly simpler than Γ_1 and Γ_2. For example, we show that the following choices of the type of sentences in the interpolant Γ are possible (see also Table 1, lines 6, 13 and 22):

- If Γ_1 consists of universal sentences and Γ_2 consists of positive sentences, then Γ consists of universally quantified disjunctions of atoms;
- If Γ_1 consists of UHC's and Γ_2 consists of positive sentences, then Γ consists of universally quantified atoms;
- If Γ_1 consists of finitary first-order sentences and Γ_2 consists of infinitary universally quantified disjunctions of atoms, then Γ consists of finitary universally quantified disjunctions of atoms.

We shall also employ our semantic technique to obtain results about Craig interpolation in institutions and about Craig-Robinson interpolation.

Motivation

Besides its intrinsic mathematical importance, Craig interpolation has applications in several areas of computer science. Such an area is formal specification theory (see [23,16]). For structured specifications [3,37], interpolation ensures a good, compositional, behavior of module semantics [3,5,32]. In choosing a logical framework for specifications, one has to find the right balance between expressive power and amenable computational aspects. Therefore, an intermediate choice between the "extremes", namely full first-order logic on the expressive side and equational logic on the computational side, might be desirable. [2] We enable such intermediate logics (e.g., the *positive*- or $(\forall\vee)$-logic) as specification frameworks, by showing that they have the interpolation property. Moreover, the very general nature of our results w.r.t. signature morphisms sometimes allows one to enrich the class of morphisms used for renaming usually up to arbitrary morphisms, freeing specifications from unnatural (but technical) constraints, like injectivity of the renaming/translation. Some technical details about the applications of our results to formal specifications can be found in Section 7.

Automatic reasoning is another area where interpolation is important and where our results contribute. There, *putting theories together* while still taking

[2] What one calls "extreme" depends of course on one's particular interest – for instance, the variable-substitution mechanism from equational logic might still be two expensive computationally for certain verification purposes, where *propositional* logic interpolation might be the desired computationally amenable extreme.

advantage, inside their union language, of their available decision procedures, relies on interpolation in a crucial way [28,30]. Moreover, interpolation provides a heuristic to "divide and conquer" a proving task: in order to show $\Gamma_1 \models_{\Sigma_1 \cup \Sigma_2} \Gamma_2$, find some Γ over the syntax $\Sigma_1 \cap \Sigma_2$ and prove the two "simpler" tasks $\Gamma_1 \models_{\Sigma_1} \Gamma$ and $\Gamma \models_{\Sigma_2} \Gamma_2$. For some simpler sub-logics of first-order logic, such as propositional calculus, where there is a finite set of semantically different sentences over any given signature, one can use interpolation also as a disproof technique: if for each $(\Sigma_1 \cap \Sigma_2)$-sentence Γ (there is only a finite number of them) at least one of $\Gamma_1 \models_{\Sigma_1} \Gamma$ or $\Gamma \models_{\Sigma_2} \Gamma_2$ fails, then $\Gamma_1 \models_{\Sigma_1 \cup \Sigma_2} \Gamma_2$ fails. The results of the present paper, although not effectively constructing interpolants, provide information about the existence of interpolants *of a certain type*, helping reducing the space of search. For instance, according to one of the cases of our main result, Theorem 5.3, the existence of a positive interpolant Γ is ensured by the fact that *either* one of Γ_1 or Γ_2 is positive (lines 2, 3 of Table 1).

The current paper is an extended version of the conference paper [31]. We have included the following additional content:

- A couple of more interpolation results about first-order sub-logics.
- A discussion of interpolation in institutions and some consequences for the higher-order and second-order logics (Section 6).
- A discussion of Craig-Robinson interpolation (Section 8).
- Full proofs and more detailed explanations and motivations for the results stated in [31].

The rest of the paper is structured into sections as follows. Section 2 introduces some technical conventions and definitions. Section 3 recalls concepts related to (many-sorted) first-order logic and interpolation, and gives examples showing failure of the interpolation property for sub-logics of first-order logic. Section 4 introduces our semantic technique for establishing interpolation in its most abstract form, in terms of operators on classes. Section 5 puts to work the concepts of Section 4 in conjunction with known axiomatizability results in order to obtain new interpolation results for sub-logics of first-order logic. Section 6 studies interpolation in institutions, again in the light of our abstract results from Section 4. Section 7 discusses potential applications of our new results to the theory of formal specifications. Section 8 deals briefly with a Craig-Robinson interpolation version of our results. Section 9 discusses related work and draws conclusions. The Appendix contains proofs that were omitted from the main text.

2 Technical Preliminaries

For simplifying the exposition, set-theoretical foundational issues are ignored in this paper.[3] Given a class $\mathcal{D}$, we let $\mathcal{P}(\mathcal{D})$ denote the collection of all subclasses of $\mathcal{D}$. For any $\mathcal{C} \in \mathcal{P}(\mathcal{D})$, let $\overline{\mathcal{C}}$ denote $\mathcal{D} \setminus \mathcal{C}$, that is, the class of all elements in $\mathcal{D}$ which are not in $\mathcal{C}$. Also, given $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{P}(\mathcal{D})$ let $[\mathcal{C}_1, \mathcal{C}_2]$ denote the collection of all classes $\mathcal{C}$ which include $\mathcal{C}_1$ and are included in $\mathcal{C}_2$. Note that $[\mathcal{C}_1, \mathcal{C}_2]$ is empty if $\mathcal{C}_1 \not\subseteq \mathcal{C}_2$.

An *operator* on class $\mathcal{D}$ is a mapping $F : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D})$. Let $Id_{\mathcal{D}}$ denote the identity operator. For any operator F on $\mathcal{D}$, let $Fixed(F)$ denote the collection of all *fixed points* of F, that is, $\mathcal{C} \in Fixed(F)$ iff $F(\mathcal{C}) = \mathcal{C}$. An operator F on $\mathcal{D}$ is a *closure operator* iff it is:

- *extensive* ($\mathcal{C} \subseteq F(\mathcal{C})$),
- *monotone* (if $\mathcal{C}_1 \subseteq \mathcal{C}_2$ then $F(\mathcal{C}_1) \subseteq F(\mathcal{C}_2)$) and
- *idempotent* ($F(F(\mathcal{C})) = F(\mathcal{C})$).

Given a binary relation $\mathcal{R}$ on $\mathcal{D}$, let $\mathcal{R}$ also denote the operator on $\mathcal{D}$ associated with $\mathcal{R}$, assigning to each $\mathcal{C} \in \mathcal{P}(\mathcal{D})$ the class of all elements from $\mathcal{D}$ in relation with elements in $\mathcal{C}$, that is, $\mathcal{R}(\mathcal{C}) = \{c' \in \mathcal{D} \mid (\exists c \in \mathcal{C})\ c\ \mathcal{R}\ c'\}$. Notice that the operator associated to a reflexive and transitive relation is a closure operator.

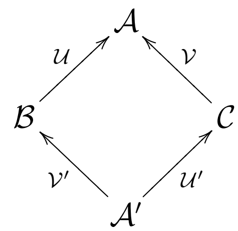
Given two classes $\mathcal{C}$ and $\mathcal{D}$ and a mapping $\mathcal{U} : \mathcal{C} \to \mathcal{D}$, we let $\mathcal{U}$ also denote the mapping $\mathcal{U} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ defined by $\mathcal{U}(\mathcal{C}') = \{\mathcal{U}(c) \mid c \in \mathcal{C}'\}$ for any $\mathcal{C}' \in \mathcal{P}(\mathcal{C})$. Also, we let $\mathcal{U}^{-1} : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$ denote the mapping defined by $\mathcal{U}^{-1}(\mathcal{D}') = \{c \in \mathcal{C} \mid \mathcal{U}(c) \in \mathcal{D}'\}$ for any $\mathcal{D}' \in \mathcal{P}(\mathcal{D})$. Given two mappings $\mathcal{U}, \mathcal{V} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$, we say that $\mathcal{U}$ *is included in* $\mathcal{V}$, written $\mathcal{U} \sqsubseteq \mathcal{V}$, iff $\mathcal{U}(\mathcal{C}') \subseteq \mathcal{V}(\mathcal{C}')$ for any $\mathcal{C}' \in \mathcal{P}(\mathcal{C})$.

We write the composition of mappings in "diagrammatic order": if $f : A \to B$ and $g : B \to C$ then $f;g$ denotes their composition, regardless of whether f and g are mappings between sets, between classes, or between collections of classes.

Definition 2.1 *We say that the mappings $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$, $\mathcal{V}'$ (between classes, like*

[3] Yet, note that references to collections of classes could be easily avoided.

in the diagram below) form a ***commutative square*** *iff* $\mathcal{V}' ; \mathcal{U} = \mathcal{U}' ; \mathcal{V}$.



A commutative square as pictured above is a ***weak amalgamation square*** *iff for all* $b \in \mathcal{B}$ *and* $c \in \mathcal{C}$ *such that* $\mathcal{U}(b) = \mathcal{V}(c)$, *there exists some* $a' \in \mathcal{A}'$ *such that* $\mathcal{V}'(a') = b$ *and* $\mathcal{U}'(a') = c$. *(We call this amalgamation square "weak" because* a' *is not required to be unique.)*

3 First-Order Logic and Classical Interpolation Revisited

A (many-sorted) first-order signature is a triple (S, F, P) consisting of a set S of sort symbols, a set F of function symbols, and a set P of relation symbols (not necessarily binary). Each function or relation symbol comes with a sequence of argument sorts, called its *arity*; function symbols come also with a *result* sort. We let $F_{w \to s}$ denote the set of function symbols with arity w and result sort s, and P_w the set of relation symbols with arity w. Given a signature Σ, the class of Σ-*models*, $Mod(\Sigma)$ consists of all first-order structures A interpreting each sort symbol s as a non-empty[4] set A_s, each function symbol σ as a function A_σ from the product of the interpretations of the argument sorts to the interpretation of the result sort, and each relation symbol π as a subset A_π of the product of the interpretations of the argument sorts.

The set of Σ-*sentences*, $Sen(\Sigma)$, consists of the usual first-order sentences, i.e., first-order formulae with no free variables, where the first-order formulae are built from equational and relational atoms by iterative applications of the logical connectives $\wedge, \vee, \neg, \Rightarrow$ and quantifiers $\forall, \exists$. The satisfaction of sentences by models ($A \models \gamma$) is the usual Tarskian notion. The satisfaction relation can be extended to a relation $\models$ between classes of models $\mathcal{M} \subseteq Mod(\Sigma)$ and sets of sentences $\Gamma \subseteq Sen(\Sigma)$: $\mathcal{M} \models \Gamma$ iff $A \models \gamma$ for all $A \in \mathcal{M}$ and $\gamma \in \Gamma$. This further induces two operators $\_^* : \mathcal{P}(Sen(\Sigma)) \to \mathcal{P}(Mod(\Sigma))$ and $\_^* : \mathcal{P}(Mod(\Sigma)) \to \mathcal{P}(Sen(\Sigma))$, defined by $\Gamma^* = \{A \mid \{A\} \models \Gamma\}$ and $\mathcal{M}^* = \{\gamma \mid \mathcal{M} \models \{\gamma\}\}$ for each $\Gamma \subseteq Sen(\Sigma)$ and $\mathcal{M} \subseteq Mod(\Sigma)$. The two operators $\_^*$ form a Galois connection between $(\mathcal{P}(Sen(\Sigma)), \subseteq)$ and $(\mathcal{P}(Mod(\Sigma)), \subseteq)$. The two composition operators $\_^* ; \_^*$ are denoted $\_^\bullet$ and are called *deduction closure* (the one on sets of sentences) and *axiomatizable hull* (the one on classes of

[4] Birkhoff-style axiomatizability, which will be used intensively in this paper, depends on the non-emptiness of carriers [35].

models). We call classes of models closed under $\_^\bullet$ *elementary classes* and sets of sentences closed under $\_^\bullet$ *theories* (terminology taken from [26]). If $\Gamma, \Gamma' \subseteq Sen(\Sigma)$, we say that Γ *semantically deduces* Γ', written $\Gamma \models \Gamma'$, iff $\Gamma^* \subseteq \Gamma'^*$.

Given two signatures $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$, a signature morphism $\phi : \Sigma \to \Sigma'$ is a triple $(\phi^{st}, \phi^{op}, \phi^{rl})$ mapping the three components in a compatible way. (When there is no danger of confusion, we let ϕ denote each of the mappings $\phi^{st}, \phi^{op}, \phi^{rl}$.) Let $\phi : \Sigma \to \Sigma'$ be a signature morphism. It has an associated sentence translation $Sen(\phi) : Sen(\Sigma) \to Sen(\Sigma')$, which renames the sorts, function-, and relation- symbols according to ϕ. Most of the times we write ϕ for $Sen(\phi)$. The reduct according to ϕ of a Σ'-model A', denoted $A'{\upharpoonright}_\phi$, is the Σ-model defined by $(A'{\upharpoonright}_\phi)_\alpha = A'_{\phi(\alpha)}$ for each sort, function, or relation symbol α in Σ. Let $Mod(\phi) : Mod(\Sigma') \to Mod(\Sigma)$ denote the mapping $A' \mapsto A'{\upharpoonright}_\phi$. Notice that *Sen* is covariant, while *Mod* is contravariant. The satisfaction relation has the important property that it is *invariant under change of notation* [21]:

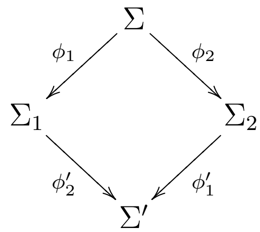
Proposition 3.1 *For each $\gamma \in Sen(\Sigma)$ and $A' \in Mod(\Sigma')$, $A' \models \phi(\gamma)$ iff $A'{\upharpoonright}_\phi \models \gamma$.*

Given two Σ-models A and B, a *morphism* $h : A \to B$ is an S-sorted function $(h_s : A_s \to B_s)_{s \in S}$ that commutes with the operations (i.e., for each function symbol σ, say of arity $s_1 \ldots s_n$ and sort s, $h_s(A_\sigma(a_1, \ldots, a_n)) = B_\sigma(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$ for all $(a_1, \ldots, a_n) \in A_{s_1} \times \ldots \times A_{s_n}$) and preserves the relations (i.e., for each predicate symbol π, say of arity $s_1 \ldots s_n$, $(a_1, \ldots, a_n) \in A_\pi$ implies $(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \in B_\pi$ for all $(a_1, \ldots, a_n) \in A_{s_1} \times \ldots \times A_{s_n}$). Models and model morphisms form a category, with composition defined as sort-wise function composition; we also let $Mod(\Sigma)$ denote it, just like the class of models. For each signature morphism $\phi : \Sigma \to \Sigma'$, the mapping $Mod(\phi)$ can be naturally extended to a functor between $Mod(\Sigma') \to Mod(\Sigma)$, defined on model morphisms similarly to the way it is defined on models. A *surjective* (*injective*) morphism is a morphism which is surjective (injective) on each sort.

Because of the weak form of commutation imposed on morphisms w.r.t. the relational part of models, relations and functions do not behave similarly along arbitrary morphisms, but only along closed ones: a morphism $h : A \to B$ is called *closed* if the relation-preservation condition holds in the "iff" form, that is, for each predicate symbol π, $(a_1, \ldots, a_n) \in A_\pi$ iff $(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \in B_\pi$. A morphism $h : A \to B$ is called *strong* if the target relations are covered through h by the source relation, that is, for each predicate symbol π and $(b_1, \ldots, b_n) \in B_\pi$, there exists $(a_1, \ldots, a_n) \in A_\pi$ such that $(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) = (b_1, \ldots, b_n)$. Closed injective morphisms and strong surjective morphisms capture the notions of embedding and homomorphic image, respectively.

We can now define the syntactic counterpart of Definition 2.1 in the case of first order logic. (We call it "syntactic" because the commutative diagram is given using morphisms of signatures.)

Definition 3.2 *A square of signature morphisms as in the diagram*



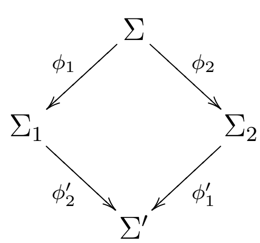
is called a **weak amalgamation square** *provided that for all models* $M_1 \in Mod(\Sigma_1)$ *and* $M_2 \in Mod(\Sigma_2)$ *with* $M_1\restriction_{\phi_1} = M_2\restriction_{\phi_2}$*, there exists a* Σ'*-model* M' *such that* $M'\restriction_{\phi_2'} = M_1$ *and* $M'\restriction_{\phi_1'} = M_2$.

Notice that a signature square is a weak amalgamation square iff its image by the functor *Mod* is a weak amalgamation square according to Definition 2.1.

Interpolation

The original formulation of interpolation [8] is in terms of signature intersections and unions, that is, w.r.t. squares which are pushouts of signature inclusions. However, subsequent advances in modularization theory [3,15,16,10,4] showed the need of arbitrary pushout squares or even weak amalgamation squares. A general formulation of interpolation is the following:

Definition 3.3 *Assume a commutative square of signature morphisms (see diagram) and two sets of sentences* $\Gamma_1 \subseteq Sen(\Sigma_1)$, $\Gamma_2 \subseteq Sen(\Sigma_2)$ *such that* $\phi_2'(\Gamma_1) \models_{\Sigma'} \phi_1'(\Gamma_2)$ *(i.e.,* Γ_1 *implies* Γ_2 *on the "union language"* Σ'*). An* ***interpolant*** *for* Γ_1 *and* Γ_2 *is a set* $\Gamma \subseteq Sen(\Sigma)$ *such that* $\Gamma_1 \models_{\Sigma_1} \phi_1(\Gamma)$ *and* $\phi_2(\Gamma) \models_{\Sigma_2} \Gamma_2$.



The following three examples show that, without further restrictions on signature morphisms, an interpolant Γ may not be found with *the same type* of sentences as Γ_1 and Γ_2, but with more general ones. In other words, there are first-order sub-logics which do not admit Craig Interpolation within themselves but in a larger (sub-)logic. The first example below shows a square in unconditional equational logic which does not admit unconditional interpolants, but admits a conditional one:

Example 3.4 *Consider the following pushout of algebraic signatures, as in [32]:* $\Sigma = (\{s\}, \{d_1, d_2 : s \to s\})$, $\Sigma_1 = (\{s\}, \{d_1, d_2, c : s \to s\})$, $\Sigma_2 = (\{s\}, \{d : s \to s\})$, $\Sigma' = (\{s\}, \{d, c : s \to s\})$, *all morphisms mapping the sort* s *to itself,* ϕ_1 *and* ϕ_2 *mapping* d_1 *and* d_2 *to themselves and to* d*, respectively,* ϕ'_2 *mapping* d_1 *and* d_2 *to* d *and* c *to itself, and* ϕ'_1 *mapping* d *to itself.*

Take $\Gamma_1 = \{(\forall x)d_2(x) = c(d_1(x)),\ (\forall x)d_1(d_2(x)) = c(d_2(x))\}$ *and* $\Gamma_2 = \{(\forall x)d(d(x)) = d(x)\}$ *to be sets of* Σ_1*-equations and of* Σ_2*-equations, respectively. It is easy to see that* Γ_1 *implies* Γ_2 *in the "union language", i.e.,* $\phi'_2(\Gamma_1) \models \phi'_1(\Gamma_2)$*. But* Γ_1 *and* Γ_2 *have no (unconditional-)equational* Σ*-interpolant, because the only equational* Σ*-consequences of* Γ_1 *are the trivial ones, of the form* $(\forall X)t = t$ *with* t *a* Σ*-term (since all the nontrivial equational* Σ_1*-consequences of* Γ_1 *contain the symbol* c*). Yet,* Γ_1 *and* Γ_2 *have a conditional-equational interpolant, e.g.,* $\{(\forall x)d_1(x) = d_2(x) \Rightarrow d_1(x) = d_1(d_1(x))\}$.

The next example shows a situation in which the interpolant cannot even be conditional-equational; it can be a more complex first-order sentence, though:

Example 3.5 *Consider the same pushout of signatures as in the previous example and take* $\Gamma_1 = \{(\forall x)d_2(x) = d_1(c(x)),\ (\forall x)d_1(d_2(x)) = d_2(c(x))\}$ *and* $\Gamma_2 = \{(\forall x)d(d(x)) = d(x)\}$*. Again,* $\phi'_2(\Gamma_1) \models \phi'_1(\Gamma_2)$*. But now* Γ_1 *and* Γ_2 *have no conditional-equational* Σ*-interpolant either, because all nontrivial conditional equations we can infer from* Γ_1 *contain* c *(to see this, think in terms of the deduction system for conditional equational logic). Nevertheless,* Γ_1 *and* Γ_2 *have a first-order interpolant, e.g.,* $\{((\forall x)d_1(x) = d_2(x)) \Rightarrow ((\forall y)d_1(y) = d_1(d_1(y)))\}$.

An obstacle to interpolation inside the desired type of sentences in the examples above is the lack of injectivity of ϕ_2 on operation symbols; injectivity on both sorts and operation symbols implies conditional equational interpolation [34].

The following example, taken from [4], shows that first-order logic does not admit interpolation either without making additional requirements on the square's morphisms.

Example 3.6 *Let* $\Sigma = (\{s_1, s_2\}, \{d_1 :\to s_1, d_2 :\to s_2\})$, $\Sigma_1 = (\{s\}, \{d_1, d_2 :\to s\})$, $\Sigma_2 = (\{s\}, \{d :\to s\})$, $\Sigma' = (\{s\}, \{d :\to s\})$*, all the morphisms mapping all sorts to* s*,* ϕ_1 *mapping* d_1 *and* d_2 *to themselves, and all the other morphisms mapping all operation symbols to* d*. In [4], it is shown that first-order interpolation does not hold in this context. For instance, let* $\Gamma_1 = \{\neg(d_1 = d_2)\}$ *and* $\Gamma_2 = \{\neg(d = d)\}$*. Then obviously* $\phi'_2(\Gamma_1) \models \phi'_1(\Gamma_2)$*, but* Γ_1 *and* Γ_2 *have* *no first-order Σ-interpolant. Indeed, assume by contradiction that there exists a set Γ of Σ-sentences such that $\Gamma_1 \models \phi_1(\Gamma)$ and $\phi_2(\Gamma) \models \Gamma_2$; let A be the Σ_1-model with $A_s = \{0,1\}$, such that $A_{d_1} = 0$ and $A_{d_2} = 1$. Let B denote $A\restriction_{\phi_1}$. We have that $B_{s_1} = B_{s_2} = \{0,1\}$, $B_{d_1} = 0$, $B_{d_2} = 1$. Because $A \models \Gamma_1$ and $\Gamma_1 \models \phi_1(\Gamma)$, it holds that $B \models \Gamma$. Define the Σ-model C to be the same as B, just that one takes $C_{d_1} = C_{d_2} = 0$. Now, C and B are isomorphic, so $C \models \Gamma$; but C admits a ϕ_2-extension D, and, because $\phi_2(\Gamma) \models \Gamma_2$, we get $D \models \Gamma_2$, which is a contradiction, since no Σ_2-model can satisfy $\neg(d = d)$. What one would need here in order to "fix" interpolation is some extension of many-sorted first-order formulae which would allow one* to equate terms of different sorts*, in the form $t_1.s_1 = t_2.s_2$; alternatively, an order-sorted second-order extension, allowing quantification over sorts, a special symbol $<$ which is to be interpreted as inclusion between sort carriers, and membership assertions $t : s$, meaning "t is of sort s" (in the spirit of [25]), would do, because we could formally state in Σ that there exists a common subsort s' of s_1 and s_2 such that $d_1 : s'$ and $\neg(d_2 : s')$.*

We shall shortly prove that for a pushout square to have first-order interpolation, it is sufficient that it has *one* of the morphisms injective *on sorts*. This is, up to our knowledge, the most general known effective criterion for a pushout to have first-order interpolation. (The same result is obtained in [6] and [17] using different techniques.)

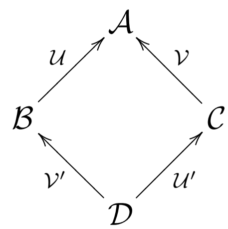
4 Semantic Interpolation

The interpolation problem, despite its syntactic nature, can be regarded *semantically*, on classes of models. Indeed, by the sentence-model duality and the satisfaction condition (Proposition 3.1), we have that:

- $\phi_2'(\Gamma_1) \models \phi_1'(\Gamma_2)$ iff $\phi_2'(\Gamma_1)^* \subseteq \phi_1'(\Gamma_2)^*$ iff $Mod(\phi_2')^{-1}(\Gamma_1^*) \subseteq Mod(\phi_1')^{-1}(\Gamma_2^*)$.
- $\Gamma_1 \models \phi_1(\Gamma)$ iff $\Gamma_1^* \subseteq \phi_1(\Gamma)^*$ iff $\Gamma_1^* \subseteq Mod(\phi_1)^{-1}(\Gamma^*)$.
- $\phi_2(\Gamma) \models \Gamma_2$ iff $\phi_2(\Gamma)^* \subseteq \Gamma_2^*$ iff $Mod(\phi_2)^{-1}(\Gamma^*) \subseteq \Gamma_2^*$.

Therefore, the interpolation property can be restated in terms of inclusions between classes of models only. If Γ is an interpolant of Γ_1 and Γ_2, we will call Γ^* a *semantic interpolant* of Γ_1^* and Γ_2^*. These suggest defining the following broader notion of "semantic interpolation":

Definition 4.1 *Consider the following commutative diagram*



(where the objects are classes and the arrows are mappings between classes) together with some $\mathcal{M} \in \mathcal{P}(\mathcal{B})$ *and* $\mathcal{N} \in \mathcal{P}(\mathcal{C})$ *such that* $\mathcal{V}'^{-1}(\mathcal{M}) \subseteq \mathcal{U}'^{-1}(\mathcal{N})$. *We say that* $\mathcal{K} \in \mathcal{P}(\mathcal{A})$ *is a* ***semantic interpolant*** *of* $\mathcal{M}$ *and* $\mathcal{N}$ *iff* $\mathcal{M} \subseteq \mathcal{U}^{-1}(\mathcal{K})$ *and* $\mathcal{V}^{-1}(\mathcal{K}) \subseteq \mathcal{N}$.

If we take $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ to be $Mod(\Sigma)$, $Mod(\Sigma_1)$, $Mod(\Sigma_2)$, $Mod(\Sigma')$ and $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$, $\mathcal{V}'$ to be $Mod(\phi_1)$, $Mod(\phi_2)$, $Mod(\phi'_1)$, $Mod(\phi'_2)$, respectively, we obtain the concrete first-order case. The connection between semantic interpolation and classical logical interpolation holds only when one considers classes which are *elementary*, i.e., specified by sets of sentences, and the interpolant is also elementary. Rephrasing the interpolation problem semantically allows us to adopt the following "divide and conquer" approach, already sketched in [32]:

(1) Find as many semantic interpolants as possible without caring whether they are axiomatizable or not (note that "axiomatizable" will mean "elementary" only within full first-order logic, but we shall consider other logics as well).
(2) Then, by imposing axiomatizability closure properties on the two starting classes of models, try to obtain a closed interpolant.

Let $\mathcal{I}(\mathcal{M}, \mathcal{N})$ denote the collection of all semantic interpolants of $\mathcal{M}$ and $\mathcal{N}$. The following gives a precise characterization of semantic interpolants together with a general condition under which they exist.

Proposition 4.2 *Under the hypothesis of Definition 4.1:*

(1) $\mathcal{I}(\mathcal{M}, \mathcal{N}) = [\mathcal{U}(\mathcal{M}), \overline{\mathcal{V}(\overline{\mathcal{N}})}]$.
(2) If the square is a weak amalgamation square then $\mathcal{I}(\mathcal{M}, \mathcal{N}) \neq \emptyset$.

Proof.

1. For any $\mathcal{K} \subseteq \mathcal{A}$, we have that $\mathcal{M} \subseteq \mathcal{U}^{-1}(\mathcal{K})$ is equivalent to $\mathcal{U}(\mathcal{M}) \subseteq \mathcal{K}$; moreover, one can see that $\mathcal{V}^{-1}(\mathcal{K}) \subseteq \mathcal{N}$ is equivalent to $\mathcal{K} \subseteq \overline{\mathcal{V}(\overline{\mathcal{N}})}$. (In categorical terms, $\mathcal{U}^{-1}$ is the left adjoint of $\mathcal{U}$ and $\mathcal{K} \mapsto \overline{\mathcal{V}(\overline{\mathcal{K}})}$ is the left adjoint of $\mathcal{V}^{-1}$.) Therefore, $\mathcal{K}$ is a semantic interpolant for $\mathcal{M}$ and $\mathcal{N}$ iff $\mathcal{U}(\mathcal{M}) \subseteq \mathcal{K} \subseteq \overline{\mathcal{V}(\overline{\mathcal{N}})}$.

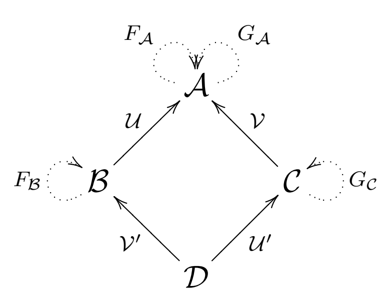
2. All we need to show is that $\mathcal{U}(\mathcal{M}) \subseteq \overline{\mathcal{V}(\overline{\mathcal{N}})}$, i.e., that for any $a \in \mathcal{U}(\mathcal{M})$, a is not an element of $\mathcal{V}(\overline{\mathcal{N}})$. Suppose it were and consider $b \in \mathcal{M}$ and $c \in \overline{\mathcal{N}}$ such that $\mathcal{U}(b) = a = \mathcal{V}(c)$. From the weak amalgamation property we deduce that there exists some $a' \in \mathcal{D}$ such that $\mathcal{V}'(a') = b$ and $\mathcal{U}'(a') = c$. Since $b \in \mathcal{M}$, it follows that $a' \in \mathcal{V}'^{-1}(\mathcal{M})$; since $\mathcal{V}'^{-1}(\mathcal{M}) \subseteq \mathcal{U}'^{-1}(\mathcal{N})$, it further follows that $a' \in \mathcal{U}'^{-1}(\mathcal{N})$, i.e., that $\mathcal{U}'(a') \in \mathcal{N}$. However, this is in contradiction with the fact that $c = \mathcal{U}'(a')$ was chosen from $\overline{\mathcal{N}}$. □

Definition 4.3 *Given two classes $\mathcal{C}$ and $\mathcal{D}$, a mapping $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ and a pair of operators $F = (F_\mathcal{C} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C}), F_\mathcal{D} : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D}))$, we say that $\mathcal{U}$* ***preserves fixed points of*** *F if $\mathcal{U}(Fixed(F_\mathcal{C})) \subseteq Fixed(F_\mathcal{D})$, that is, for any fixed point of $F_\mathcal{C}$ we obtain through $\mathcal{U}$ a fixed point of $F_\mathcal{D}$; also we say that $\mathcal{U}$* ***lifts*** *F if $F_\mathcal{D} ; \mathcal{U}^{-1} \sqsubseteq \mathcal{U}^{-1} ; F_\mathcal{C}$, that is, for any $\mathcal{D}' \in \mathcal{P}(\mathcal{D})$ and any $c \in \mathcal{C}$, if $\mathcal{U}(c) \in F_\mathcal{D}(\mathcal{D}')$ then $c \in F_\mathcal{C}(\mathcal{U}^{-1}(\mathcal{D}'))$.*

Preservation of fixed points of operators is a property frequently encountered in logic and algebra. Assume $\mathcal{U}$ is the reduct functor $Mod(\phi)$ for some signature morphism ϕ. If F is closure under ultraproducts, preserving its fixed points means commuting with ultraproducts; if F is closure under quotients, $\mathcal{U}$ preserving its fixed points means being able to expand each quotient of some model $\mathcal{U}(M)$ to a quotient of M. "Lifting" is a somehow less intuitive property, but in cases of operators given by relations it tends to be dual to preservation of fixed points (see Proposition 4.5 below). For example, if $\mathcal{U}$ is as above and if F is closure under quotients, $\mathcal{U}$ lifting F means being able to expand each model for which $\mathcal{U}(M)$ is a quotient to a model for which M is a quotient. Our main results in Section 5 will employ lifting and fixed-point preserving properties for a large variety of concrete operators on classes of models.

The following theorem is at the heart of all our subsequent results. It gives general criteria under which a weak amalgamation square admits semantic interpolants *closed* under some generic operators.

Theorem 4.4 *Consider a weak amalgamation square as in the diagram below and two pairs of operators $F = (F_\mathcal{B} : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B}), F_\mathcal{A} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}))$ and $G = (G_\mathcal{C} : \mathcal{P}(C) \to \mathcal{P}(C), G_\mathcal{A} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}))$ such that:*



(1) $F_{\mathcal{A}} \,;G_{\mathcal{A}} \,;F_{\mathcal{A}} = F_{\mathcal{A}} \,;G_{\mathcal{A}}$.
(2) $G_{\mathcal{C}}$ *and* $G_{\mathcal{A}}$ *are closure operators.*
(3) $\mathcal{U}$ *preserves fixed points of* F.
(4) $\mathcal{V}$ *lifts* G.

Then for each $\mathcal{M} \in Fixed(F_{\mathcal{B}})$ *and* $\mathcal{N} \in Fixed(G_{\mathcal{C}})$ *such that* $\mathcal{V}'^{-1}(\mathcal{M}) \subseteq \mathcal{U}'^{-1}(\mathcal{N})$, $\mathcal{M}$ *and* $\mathcal{N}$ *have a semantic interpolant* $\mathcal{K}$ *in* $Fixed(F_{\mathcal{A}}) \cap Fixed(G_{\mathcal{A}})$.

Proof.

Take $\mathcal{K} = G_{\mathcal{A}}(\mathcal{U}(\mathcal{M}))$. Let us first show that $\mathcal{K} \in Fixed(F_{\mathcal{A}}) \cap Fixed(G_{\mathcal{A}})$. We have that $\mathcal{K} \in Fixed(G_{\mathcal{A}})$, since $G_{\mathcal{A}}$ is idempotent. Also, since $\mathcal{M} \in Fixed(F_{\mathcal{B}})$ and $\mathcal{U}$ preserves fixed points of F, we have that $\mathcal{U}(\mathcal{M}) \in Fixed(F_{\mathcal{A}})$. Therefore $F_{\mathcal{A}}(\mathcal{K}) = F_{\mathcal{A}}(G_{\mathcal{A}}(\mathcal{U}(\mathcal{M}))) = F_{\mathcal{A}}(G_{\mathcal{A}}(F_{\mathcal{A}}(\mathcal{U}(\mathcal{M})))) = G_{\mathcal{A}}(F_{\mathcal{A}}(\mathcal{U}(\mathcal{M}))) = G_{\mathcal{A}}(\mathcal{U}(\mathcal{M})) = \mathcal{K}$, that is, $\mathcal{K} \in Fixed(F_{\mathcal{A}})$.

Let us next show that $\mathcal{K}$ is a semantic interpolant of $\mathcal{M}$ and $\mathcal{N}$. Since $G_{\mathcal{A}}$ is extensive, $\mathcal{U}(\mathcal{M}) \subseteq G_{\mathcal{A}}(\mathcal{U}(\mathcal{M}))$, whence $\mathcal{M} \subseteq \mathcal{U}^{-1}(\mathcal{K})$. Using that $\mathcal{V}$ lifts G, we obtain that $\mathcal{V}^{-1}(\mathcal{K}) = \mathcal{V}^{-1}(G_{\mathcal{A}}(\mathcal{U}(\mathcal{M}))) \subseteq G_{\mathcal{C}}(\mathcal{V}^{-1}(\mathcal{U}(\mathcal{M})))$. From Proposition 4.2 we know that $\mathcal{U}(\mathcal{M})$ is a semantic interpolant of $\mathcal{M}$ and $\mathcal{N}$, so $\mathcal{V}^{-1}(\mathcal{U}(\mathcal{M})) \subseteq \mathcal{N}$. Using that $G_{\mathcal{C}}$ is monotone, we get that $G_{\mathcal{C}}(\mathcal{V}^{-1}(\mathcal{U}(\mathcal{M}))) \subseteq G_{\mathcal{C}}(\mathcal{N}) = \mathcal{N}$. Thus $\mathcal{V}^{-1}(\mathcal{K}) \subseteq \mathcal{N}$. We obtained that $\mathcal{K}$ is also a semantic interpolant of $\mathcal{M}$ and $\mathcal{N}$. $\Box$

The operators above will be conveniently chosen in the next section to be closure operators characterizing axiomatizable classes of models. The two types of axiomatizability that we consider as attached to F and G need not be the same, i.e., the classes $\mathcal{M}$ and $\mathcal{N}$ need not be axiomatizable by the same type of first-order sentences. And in the most fortunate cases, as we shall see below, the interpolant is able to capture and even strengthen the properties of both classes.

The remaining of this section provides some general properties of operators (w.r.t. their generating relations, composition, preservation and lifting), that will be used in our subsequent interpolation results. The rather straightforward proofs of these properties are exiled into the appendix.

Operators given by relations

The intuition for the word "lifts" used in Definition 4.3 comes from the case when the operators $F_{\mathcal{C}}$ and $F_{\mathcal{D}}$ are given by binary relations. Because throughout the paper many of the employed operators are associated to (reflexive and transitive) relations, let us next give an easy criterion for a mapping to lift/[preserve fixed points of] such an operator.

Proposition 4.5 *Consider two classes $\mathcal{C}$ and $\mathcal{D}$, a mapping $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ and a pair of relations $R = (R_\mathcal{C}, R_\mathcal{D})$, with $R_\mathcal{C} \subseteq \mathcal{C} \times \mathcal{C}$ and $R_\mathcal{D} \subseteq \mathcal{D} \times \mathcal{D}$.*[5] *Then the following hold:*

(1) $\mathcal{U}$ lifts R if and only if for any elements $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $dR_\mathcal{D}\mathcal{U}(c)$, there exists $c' \in \mathcal{C}$ such that $\mathcal{U}(c') = d$ and $c'R_\mathcal{C}c$.
(2) Suppose $R_\mathcal{C}$ is reflexive and transitive. Then $\mathcal{U}$ preserves fixed points of R if and only if for all elements $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $\mathcal{U}(c)R_\mathcal{D}d$, there exists $c' \in \mathcal{C}$ such that $\mathcal{U}(c') = d$ and $cR_\mathcal{C}c'$.
(3) Suppose $R_\mathcal{C}$ is reflexive and transitive. Then $\mathcal{U}$ preserves fixed points of R if and only if $\mathcal{U}$ lifts $(R_\mathcal{C}^{-1}, R_\mathcal{D}^{-1})$.
(4) Let R^+ be the pair $(R_\mathcal{C}^+, R_\mathcal{D}^+)$, where $R_\mathcal{C}^+$ and $R_\mathcal{D}^+$ are the transitive closures of $R_\mathcal{C}$ and $R_\mathcal{D}$ respectively. Then $\mathcal{U}$ lifts R^+ if $\mathcal{U}$ lifts R.

Composed operators

Since the closure operators coming from axiomatizability are in general compositions of other closure operators, we shall need the following results in order to break composed operators into their components that can be treated separately.

Proposition 4.6 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be classes and consider the following diagram:*

$$\mathcal{P}(\mathcal{A}) \xrightarrow{V} \mathcal{P}(\mathcal{B}) \underset{U'}{\overset{U}{\rightrightarrows}} \mathcal{P}(\mathcal{C}) \xrightarrow{V'} \mathcal{P}(\mathcal{D})$$

such that $U \sqsubseteq U'$. Then:

(1) $V\,; U \sqsubseteq V\,; U'$.
(2) V' monotone implies that $U\,; V' \sqsubseteq U'\,; V'$.

Proposition 4.7 *Let F and G be operators on the same class $\mathcal{D}$ such that F is a closure operator. The following hold:*

(1) If $G\,; F \sqsubseteq F\,; G$ then $F\,; G\,; F = F\,; G$.
(2) If G is also a closure operator, then $F\,; G$ is a closure operator iff $G\,; F \sqsubseteq F\,; G$.

Proposition 4.8 *Consider two classes $\mathcal{C}$ and $\mathcal{D}$, a mapping $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ and two pairs of operators $F = (F_\mathcal{C}, F_\mathcal{D})$ and $G = (G_\mathcal{C}, G_\mathcal{D})$. Then the following hold:*

(1) If $G_\mathcal{C}$ is monotone and $\mathcal{U}$ lifts F and G, then $\mathcal{U}$ also lifts $(F_\mathcal{C}\,; G_\mathcal{C}, F_\mathcal{D}\,; G_\mathcal{D})$.

[5] Recall that $R_\mathcal{C}$ and $R_\mathcal{D}$ also denote the induced operators.

(2) If $F_{\mathcal{C}}$ and $G_{\mathcal{C}}$ are extensive, $G_{\mathcal{C}}$ is monotone and $\mathcal{U}$ preserves fixed points of F and G, then $\mathcal{U}$ also preserves fixed points of $(F_{\mathcal{C}}\,; G_{\mathcal{C}}, F_{\mathcal{D}}\,; G_{\mathcal{D}})$.
(3) If $F_{\mathcal{C}}$ and $G_{\mathcal{C}}$ are extensive and $F_{\mathcal{C}}; G_{\mathcal{C}}$ is idempotent, then $F_{\mathcal{C}}; G_{\mathcal{C}}; F_{\mathcal{C}} = F_{\mathcal{C}}; G_{\mathcal{C}}$.

5 New Interpolation Results for Combinations of First-Order Sub-Logics

In this section we give a series of novel interpolation results for various types of first-order sentences.

Recall the following types of first-order sentences:

- $\mathcal{FO}$: first-order sentences.
- $\mathcal{P}os$: *positive* sentences, that is, constructed inductively from atomic formulae by means of any first-order constructs, except negation.
- $\forall$: sentences $(\forall x_1, x_2, \ldots, x_k)e$, where e is a quantifier free formula.
- $\exists$: sentences $(\exists x_1, x_2, \ldots, x_k)e$, where e is a quantifier free formula.
- $\mathcal{UH}$, *universal Horn clauses*, that is, $(\forall x_1, x_2, \ldots, x_k)(e_1 \wedge e_2 \ldots \wedge e_p) \Rightarrow e$, with e_i, e atomic formulae.
- $\mathcal{UA}$, *universal atoms*, that is, $(\forall x_1, x_2, \ldots x_k)e$, where e is an atomic formula.
- $\forall\vee$, universally quantified disjunctions of atoms, i.e., $(\forall x_1, x_2, \ldots, x_k)(e_1 \vee e_2 \ldots \vee e_p)$ where e_i are atomic formulae.
- $\mathcal{FO}_\infty$, $\mathcal{UH}_\infty$, $\forall\vee_\infty$, the infinitary extensions of $\mathcal{FO}$, $\mathcal{UH}$, $\forall\vee$, respectively; in the former case, infinite conjunction and disjunction are allowed; in the latter two cases, $e_1 \wedge e_2 \ldots \wedge e_p$ and $e_1 \vee e_2 \ldots \vee e_p$ are replaced by any possibly infinite sentence- conjunction and disjunction respectively.
- $\square$, sentences of the form $(\exists x_1)(\forall y_1^1, y_1^2, \ldots, y_1^{p_1}) \ldots (\exists x_k)(\forall y_k^1, y_k^2, \ldots, y_k^{p_k}) \bigwedge_{u=1}^{r} \bigvee_{v=1}^{s_u} e_{u,v}$, where $k, r, p_i, s_u \in \mathbb{N}$, and each $e_{u,v}$ is either atomic, or of the form $\neg\sigma(y_i^1, \ldots, y_i^{p_i-1}) = y_{p_i}^i$, or of the form $\neg\pi(y_i^1, \ldots, y_i^{p_i})$.

We next recall some basic model theoretic notions, such as submodel, product, filter, filtered product, ultrafilter, ultraproduct, ultrapower and ultraradical; the reader is referred to [7,26] for more intuition and discussion on these notions.

Let (S, F, P) be a first-order signature.

- A model B is a *submodel* of A if the carrier set of B is included in that of A and operations and relations in B are interpreted as the restriction of those in A to the carrier of B.

- Given a family $(A_i)_{i\in I}$ of S-sorted sets (where each A_i is thus a family $(A_{i,s})_{s\in S}$), the *product* of this family, denoted $\prod_{i\in I} A_i$, is the S-sorted set $(B_s)_{s\in S}$, where each B_s is $\{(a_{i,s})_{i\in I} : \forall i.a_{i,s} \in A_{i,s}\}$. The product of a family of *models* $(A_i)_{i\in I}$ is the usual set-theoretic one, canonically constructed on the product of the (S-sorted) carrier sets A_i.
- A *filter* $\mathcal{F}$ on I, $\mathcal{F} \subseteq \mathcal{P}(I)$, is characterized by the following: $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ is closed under (finite) intersection (if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$) and if $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.
- Given a family $(A_i)_{i\in I}$ of models and a filter $\mathcal{F}$ on I, the *filtered product* of $(A_i)_{i\in I}$ over $\mathcal{F}$, denoted $\prod_{\mathcal{F}} A_i$, has the carrier $B = \prod_{i\in I} A_i/\equiv$, where $\equiv = (\equiv_s)_{s\in S}$ is the S-sorted equivalence given by: $(a_{i,s})_{i\in I} \equiv_s (b_{i,s})_{i\in I}$ iff $\{i \in I \mid a_{i,s} = b_{i,s}\} \in \mathcal{F}$. Operations and relations are defined by $B_\sigma((a_i^1)_{i\in I}/\equiv_{s_1}, \ldots, (a_i^n)_{i\in I}/\equiv_{s_n}) = (B_\sigma(a_i^1, \ldots, a_i^n))_{i\in I}/\equiv_s$ for $\sigma \in F_{s_1\ldots s_n,s}$ and $B_\pi = \{((a_i^1)_{i\in I}/\equiv_{s_1}, \ldots, (a_i^n)_{i\in I}/\equiv_{s_n}) \mid \{i \in I \mid (a_i^1, \ldots, a_i^n) \in A_{i\pi}\} \in \mathcal{F}\}$ for $\pi \in P_{s_1\ldots s_n}$.
- An *ultrafilter* is a filter $\mathcal{F}$ such that if $A \cup B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.
- If $\mathcal{F}$ is an *ultrafilter*, then $\prod_{\mathcal{F}} A_i$ is said to be an *ultraproduct*. If, moreover, all A_i's are equal to some model A, then $\prod_{\mathcal{F}} A_i$ is written $A^I/_{\mathcal{F}}$ and said to be the *ultrapower* of A over $\mathcal{F}$. In this latter case, A is said to be an *ultraradical* of $A^I/_{\mathcal{F}}$.

The main point to notice in the above definitions is that they are *set-theoretical*, and not *categorical*, defining objects that are genuinely unique, and not just unique up to isomorphism. In order to emphasize this fact, we shall sometimes refer to the above defined products/filtered products/ultraproducts as "canonical products/filtered products/ultraproducts".

Consider the following binary relations on Σ-models:

- A ***S*** B iff B is isomorphic to a *submodel* of A.
- A ***Ext*** B iff B is isomorphic to an *extension* of A, i.e., to a model C such that A is a submodel of C.
- A ***H*** B iff there exists a *surjective* morphism between A and B.
- A ***Hs*** B iff there exists a *strong surjective* morphism between A and B.
- A ***Ur*** B iff A and B are isomorphic or B is an *ultraradical* of a model isomorphic to A (in other words, if A is either isomorphic to B or isomorphic to an *ultrapower* of B).

Recall that any binary relation, in particular the ones on $Mod(\Sigma)$ above, has an associated operator bearing the same name. Besides these operators, we shall also consider the operators P, Fp, and Up on $Mod(\Sigma)$ defined below:

- $P(\mathcal{M}) = \mathcal{M} \cup \{$all *products* of models in $\mathcal{M}\}$.
- $Fp(\mathcal{M}) = \mathcal{M} \cup \{$all *filtered products* of models in $\mathcal{M}\}$.

- $Up(\mathcal{M}) = \mathcal{M} \cup \{$all *ultraproducts* of models in $\mathcal{M}\}$.

The next proposition collects some known axiomatizability results. For details, the reader is referred to [7] (Section 5.2), [26] (Sections 25 and 26), [1], [29], and [10]. Below, e.g., the pair $(\mathcal{UA}, \{S, H, P\})$ corresponds to the famous Birkhoff Theorem (a class of algebras is equationally axiomatizable iff it is closed under subalgebras, homomorphic images, and products) and the pair $(\mathcal{FO}, \{Up, Ur\})$ corresponds to the Keisler-Shelah Theorem (a class of first-order models is elementary iff it is closed under ultraproducts and ultraradicals).

Proposition 5.1 *If the pair (T, Ops), consisting of a type T of Σ-sentences and a set Ops of operators on $Mod(\Sigma)$, is one of $(\mathcal{FO}, \{Up, Ur\})$, $(\mathcal{P}os, \{Up, Ur, H\})$, $(\forall, \{S, Up\})$, $(\exists, \{Ext, Up, Ur\})$, $(\mathcal{UH}, \{S, Fp\})$, $(\mathcal{UA}, \{S, H, P\})$, $(\forall\vee, \{Hs, S, Up\})$, $(\Box, \{Hs, Up\})$, $(\mathcal{UH}_\infty, \{S, P\})$, $(\forall\vee_\infty, \{Hs, S\})$, then $\mathcal{M} \subseteq Mod(\Sigma)$ is of the form Γ^* with $\Gamma \subseteq T$ iff $\mathcal{M}$ is a fixed point of all the operators in Ops.*

Consider the following syntactic properties for a morphism $\phi : \Sigma \to \Sigma'$, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$:

(IS) ϕ is ***injective on sorts***.
(IR) ϕ is ***injective on relation symbols***.
(I) ϕ is ***injective*** on sorts, operation- and relation- symbols
(RS) there are no operation symbols in $F' \setminus \phi(F)$, having the ***result sort*** in $\phi(S)$.

The next proposition, whose proof can be found in the appendix, relates syntactic properties of signature morphisms with semantic lifting and preserving properties of the corresponding operator on models.

Proposition 5.2 *For each signature morphism $\phi : \Sigma \to \Sigma'$,*

(1) $Mod(\phi)$ preserves fixed points of P, Fp, Up.
(2) (I) $\Rightarrow$ *$Mod(\phi)$ lifts S, H, Hs and preserves fixed points of Ext* [10].
(3) (IS) *and* (RS) $\Rightarrow$ *$Mod(\phi)$ preserves fixed points of S, Hs, and lifts Ext.*
(4) (IS), (IR) *and* (RS) $\Rightarrow$ *$Mod(\phi)$ preserves fixed points of H.*
(5) (IS) $\Rightarrow$ *$Mod(\phi)$ lifts Ur.*

Table 1 lists interpolation results for various types of sentences. It should be read as: given a weak amalgamation square of signatures as in Definition 3.3 and $\Gamma_1 \subseteq Sen(\Sigma_1)$, $\Gamma_2 \subseteq Sen(\Sigma_2)$, if Γ_1 and Γ_2 are sets of sentences of the indicated types such that $\phi_2'(\Gamma_1) \models \phi_1'(\Gamma_2)$, then they have an interpolant Γ of the indicated type; the semantic conditions under which this situation holds are given in the $Mod(\phi_1)$- and $Mod(\phi_2)$- columns of the table, with the meaning that $Mod(\phi_1)$ *preserves* fixed points of the indicated operator and $Mod(\phi_2)$ *lifts* the indicated operator. (*Id* is the identity operator.) These semantic conditions are implied by the syntactic conditions listed in the ϕ_1-

	Γ_1 Type	Γ_2 Type	Γ Type	$Mod(\phi_1)$ preserves	$Mod(\phi_2)$ lifts	ϕ_1	ϕ_2
1	$\mathcal{FO}$	$\mathcal{FO}$	$\mathcal{FO}$	*Up*	*Ur*	any	(IS)
2	$\mathcal{FO}$	$\mathcal{P}os$	$\mathcal{P}os$	*Up*	*H* ; *Ur*	any	(I)
3	$\mathcal{P}os$	$\mathcal{FO}$	$\mathcal{P}os$	*Up* ; *H*	*Ur*	(IS), (IR), (RS)	(IS)
4	$\mathcal{FO}$	$\forall$	$\forall$	*Up*	*S*	any	(I)
5	$\forall$	$\mathcal{FO}$	$\forall$	*Up* ; *S*	*Id*	(IS), (RS)	any
6	$\forall$	$\mathcal{P}os$	$\forall\vee$	*Up* ; *S*	*Hs*	(IS), (RS)	(I)
7	$\mathcal{FO}$	$\exists$	$\exists$	*Up*	*Ext*; *Ur*	any	(IS), (RS)
8	$\exists$	$\mathcal{FO}$	$\exists$	*Up* ; *Ext*	*Ur*	(I)	(IS)
9	$\mathcal{FO}$	$\mathcal{UH}$	$\mathcal{UH}$	*Fp*	*S*	any	(I)
10	$\mathcal{UH}$	$\mathcal{FO}$	$\mathcal{UH}$	*Fp* ; *S*	*Id*	(IS), (RS)	any
11	$\mathcal{UH}$	$\mathcal{UA}$	$\mathcal{UA}$	*P*	*S* ; *H*	any	(I)
12	$\mathcal{UA}$	$\mathcal{FO}$	$\mathcal{UA}$	*P* ; *S* ; *H*	*Id*	(IS), (IR), (RS)	any
13	$\mathcal{UH}$	$\mathcal{P}os$	$\mathcal{UA}$	*P* ; *S*	*H*	(IS),(RS)	(I)
14	$\mathcal{FO}$	$\forall\vee$	$\forall\vee$	*Up*	*S* ; *Hs*	any	(I)
15	$\forall\vee$	$\mathcal{FO}$	$\forall\vee$	*Up* ; *S* ; *Hs*	*Id*	(IS), (RS)	any
16	$\mathcal{FO}$	$\Box$	$\Box$	*Up*	*Hs*	any	(I)
17	$\Box$	$\mathcal{FO}$	$\Box$	*Up* ; *Hs*	*Id*	(IS), (RS)	any
18	$\mathcal{UH}_\infty$	$\mathcal{UA}$	$\mathcal{UA}$	*P*	*S* ; *H*	any	(I)
19	$\mathcal{UH}_\infty$	$\mathcal{FO}_\infty$	$\mathcal{UH}_\infty$	*P* ; *S*	*Id*	(IS), (RS)	any
20	$\mathcal{FO}_\infty$	$\forall\vee_\infty$	$\forall\vee_\infty$	*Id*	*S* ; *Hs*	any	(I)
21	$\forall\vee_\infty$	$\mathcal{FO}_\infty$	$\forall\vee_\infty$	*S* ; *Hs*	*Id*	(IS), (RS)	any
22	$\mathcal{FO}$	$\forall\vee_\infty$	$\forall\vee$	*Up*	*S* ; *Hs*	any	(I)

Table 1
A summary of the Craig interpolation results for first-order sub-logics.

and ϕ_2- columns; "any" means that no restriction is posed on the signature morphism.

Theorem 5.3 *The results stated in Table 1 hold, i.e., in each of the 22 cases, if ϕ_1 and ϕ_2 satisfy the indicated properties, Γ_1 and Γ_2 have the indicated types and $\phi'_2(\Gamma_1) \models \phi'_1(\Gamma_2)$, then there exists an interpolant Γ of the indicated type.*

Proof.

Let F and G be the operators in the $Mod(\phi_1)$- and $Mod(\phi_2)$- column respectively, and let T, T_1, T_2 be the types of sentences listed on the same line in the columns of $\Gamma, \Gamma_1, \Gamma_2$ respectively. Notice that, by language abuse, we let F denote either of the two operators having the same shape, $F_\Sigma : \mathcal{P}(Mod(\Sigma)) \to \mathcal{P}(Mod(\Sigma))$ and $F_{\Sigma_1} : \mathcal{P}(Mod(\Sigma_1)) \to \mathcal{P}(Mod(\Sigma_1))$. Likewise, G denotes either of the two operators having the same shape, $G_\Sigma : \mathcal{P}(Mod(\Sigma)) \to \mathcal{P}(Mod(\Sigma))$ and $G_{\Sigma_2} : \mathcal{P}(Mod(\Sigma_2)) \to \mathcal{P}(Mod(\Sigma_2))$. Thus F has the form $F_1; \ldots; F_k$ and G has the form $G_1; \ldots; G_l$, where $k, l \in \{1, 2, 3\}$. (For example, considering row 11 in Table 1, we have that $k = 1$, $F = F_1 = P$, $l = 2$ and $G = G_1; G_2$, where $G_1 = S$ and $G_2 = H$.) Because, in each case in the table, all F_i's are extensive and monotone, we obtain that a set of models is a fixed point of F iff it is a fixed point of all of $F_1, \ldots, F_k$. And likewise for G and $G_1, \ldots, G_l$.

By Proposition 5.1, ${\Gamma_1}^*$ is a fixed point of all of $F_1, \ldots, F_k$, hence of F, and ${\Gamma_2}^*$ is a fixed point of all of $G_1, \ldots, G_l$, hence of G. If we manage to show that the hypotheses (1)-(4) of Theorem 4.4 are satisfied for $F, G, \mathcal{U} = Mod(\phi_1), \mathcal{V} = Mod(\phi_2), \mathcal{U}' = Mod(\phi_1')$ and $\mathcal{V}' = Mod(\phi_2')$, then we obtain, by the mentioned theorem, a semantic interpolant $\mathcal{K}$ for ${\Gamma_1}^*$ and ${\Gamma_2}^*$ which is a fixed point of F and G, hence a fixed point of all of $F_1, \ldots, F_k, G_1, \ldots, G_l$. Finally, applying Proposition 5.1, we obtain that there exists a set of sentences $\Gamma \subseteq Sen(\Sigma)$ of type indicated in the table such that $\Gamma^* = \mathcal{K}$, making Γ the desired (syntactic) interpolant for Γ_1 and Γ_2.

So we are left with verifying the hypotheses of Theorem 4.4.

We check hypothesis 1. All the operators $F_1, \ldots, F_k, G_1, \ldots, G_l$ are extensive and monotone and moreover, by Proposition 5.2, $Mod(\phi_1)$ preserves the fixed points of each of $F_1, \ldots, F_k$ and $Mod(\phi_2)$ lifts each of $G_1, \ldots, G_l$. Thus, according to Proposition 4.8.(3), all we need to check is that $F; G$ is idempotent. It is actually the case that $F; G$ is a closure operator, because it coincides with the axiomatizability hull operator corresponding to the type T. (Indeed, by Proposition 5.1, a class of models is T-axiomatizable iff it is a fixed point of all of $F_1, \ldots, F_k, G_1, \ldots, G_l$, i.e., a fixed point of F and G.)

We check hypothesis 2. Well-known closure operators are Id, S, H, Hs (obviously) and Ur (see [7]). Moreover, $H\,; Ur$, $Ext\,; Ur$, $S\,; H$, and $S\,; Hs$ are closure operators by Proposition 4.6.(2) because their components are closure operators and because $Ur\,; H \sqsubseteq H\,; Ur$, $Ur\,; Ext \sqsubseteq Ext\,; Ur$, $H\,; S \sqsubseteq S\,; H$, and $Hs\,; S \sqsubseteq S\,; Hs$. Indeed, the first two equalities hold because, for a model A, an ultrapower $A^I/\mathcal{F}$ and a homomorphic image (extension) B of A, $B^I/\mathcal{F}$ is an homomorphic image (extension) of $A^I/\mathcal{F}$. The last two equalities hold because, if $h : B \to C$ is a (strong) surjective morphism and A is a submodel of C, then $h^{-1}(A)$, which is a submodel of B with induced operations and relations, yields a restriction-corestriction of h to $h^{-1}(A) \to A$ which is also a (strong) surjective morphism.

Finally, we check hypotheses 3 and 4, i.e., prove that if ϕ_1 and ϕ_2 are as indicated then $Mod(\phi_1)$ preserves fixed points of F and $Mod(\phi_2)$ lifts G. By Proposition 5.2, $Mod(\phi_1)$ preserves fixed points of all of $F_1, \ldots, F_k$ and $Mod(\phi_2)$ lifts all of $G_1, \ldots, G_l$. Moreover, since all the involved operators are monotone and extensive, we can apply Proposition 4.8.(1,2) to obtain that $Mod(\phi_1)$ preserves fixed points of F and $Mod(\phi_2)$ lifts G. $\Box$

Let us discuss the results listed in the table above. The syntactic conditions on signature morphisms are in many cases weaker than, or equal to, injectivity (I). In fact, if we consider only relational languages, i.e., without operation symbols, *all* the conditions are so (because (RS) becomes vacuous). As for operation symbols, it is interesting to note that (RS) comprises the principle of data encapsulation expressed in algebraic terms [19]. As also suggested by the examples in Section 3, it seems that the degree of generality that one can allow on signature morphisms increases with the expressive power of a logic. For instance, line 1 says that first-order interpolation holds whenever the right-hand morphism is injective on sorts (and, in fact, since in full first-order logic Craig interpolation is equivalent to the symmetrical property of Robinson consistency,[6] *either one* of the morphisms being injective on sorts would do). On the other hand, universal Horn clauses (lines 9 and 10), and then universal atoms (lines 11, 12, 13) require stronger and stronger assumptions on the signature morphisms. Our results say more than interpolation within a certain type T of sentences: the interpolant has type T provided *one* of the starting sets has type T. Particularly interesting results are listed in lines 6, 13, and 22, where the interpolant strictly "improves" the type of both sides.

Regarding the finiteness of the interpolant Γ, as noted in [10], it is easy to see that if Γ_2 is finite, by the compactness of first-order logic, the interpolant Γ can be chosen to also be finite in our cases of finitary first-order sub-logics. On the other hand, the finiteness of Γ_1 does not necessarily imply the finite axiomatizability of Γ^*. Indeed, assume that $\Sigma = \Sigma_2 \subseteq \Sigma_1$, ϕ_1 is the inclusion of signatures, and ϕ_2 the identity. Then $\Sigma' = \Sigma_1$, $\phi'_1 = \phi_1$, and $\phi'_2 = \phi_2$. Thus the finite interpolation problem comes to the following: assuming $\Gamma_1 \models \Gamma_2$ in Σ_1, find a finite $\Gamma \subseteq Sen(\Sigma)$ such that $\Gamma_1 \models \Gamma \models \Gamma_2$; in other words, prove that there exists a finite subset Δ_1 of ${\Gamma_1}^\bullet$ consisting of Σ-sentences such that $\Delta_1 \models \Gamma_2$ in Σ. But this cannot be always achieved, as shown by the case where Γ_2 is a Σ-theory ($\bullet$-closed set of sentences), finitely axiomatizable over the extended signature Σ_1 by Γ_1, but not finitely axiomatizable over Σ. (Such a theory is known to exist by a famous theorem of Kleene.) In our model-theoretical approach, the impossibility of relating the finiteness of Γ_1 to that of Γ is illustrated by the fact that the operator of taking ultraproduct components (classically related to finite axiomatizability [7]) is not preserved by reduct functors (but it is lifted by them).

[6] This is not true however for our examples of first-order sub-logics.

6 Interpolation in Institutions

Institutions were introduced in [20] with the original goal of providing a logic-independent framework for algebraic specifications of computer science systems. However, by isolating the essence of a logical system in the abstract *satisfaction relation*, institutions proved to be suitable for developing what was coined as "abstract abstract model theory" [37] (so to distinguish this approach from the less abstract "abstract model theory" as presented in [2]) - see [27] for an up-to-date discussion on institutions as abstract logics. Here we compare our set-theoretical interpolation result in Theorem 4.4 with another very generic result obtained in [10] in the institutional setting, showing that the latter follows from the former. Then we prove another institutional corollary of Theorem 4.4, showing that interpolation in a logic is brought by expressive enough universal quantification. Interpolation results for (language-finitary variants of) second- and higher-order logics (with standard models) are instances of this corollary.

An institution [20,21] consists of:

(1) A category *Sign*, whose objects are called *signatures*.
(2) A functor $Sen : Sign \to Set$, providing for each signature Σ a set whose elements are called *(Σ-)sentences*.
(3) A functor $Mod : Sign \to Cat^{op}$, providing for each signature Σ a category whose objects are called *(Σ-)models* and whose arrows are called *(Σ-)morphisms*.
(4) A relation $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |Sign|$, called *(Σ-) satisfaction*, such that for each morphism $\phi : \Sigma \to \Sigma'$ in *Sign*, the *satisfaction condition*

$$M' \models_{\Sigma'} Sen(\phi)(e) \text{ iff } Mod(\phi)(M') \models_\Sigma e$$

holds for all $M' \in |Mod(\Sigma')|$ and $e \in Sen(\Sigma)$. Following the usual notational conventions, we sometimes let $\_\!\upharpoonright_\phi$ denote the reduct functor $Mod(\phi)$ and let ϕ denote the sentence translation $Sen(\phi)$.

In Section 3, when we defined the signatures and models (together with their morphisms) and the sentences of first-order logic and when we talked about the invariance of satisfaction under change of notation, we were actually describing the institution of first-order logic [21]. All the syntactic-detail free concepts defined there for first-order logic (like deduction closure, axiomatizability hull, theories, elementary classes) make sense in the abstract framework of institutions as well. Other examples of institutions include the second-order and higher-order logics, which will be discussed in Section 6.2.

Birkhoff institutions were introduced in [10] as a common framework for logics exhibiting Birkhoff style axiomatizability properties, in that their semantic consequence operator is expressible in terms of standard operators on classes of models. A Birkhoff institution is an institution $(\mathit{Sign}, \mathit{Sen}, \mathit{Mod}, \models)$ such that the category $\mathit{Mod}(\Sigma)$ has products and directed colimits (hence filtered products) for each signature Σ, together with

- a class $I\!F$ of pairs $(I, \mathcal{F})$, where I is a set and $\mathcal{F}$ a filter on I such that $(\{*\}, \{\{*\}\})$ is in $I\!F$, where $\{*\}$ is a singleton set,
- a binary relation $B_\Sigma \subseteq |\mathit{Mod}(\Sigma)| \times |\mathit{Mod}(\Sigma)|$ for each signature Σ that includes the model-isomorphism relation,

such that $\mathcal{M}^\bullet = B_\Sigma^{-1}(I\!F\mathcal{M})$ for each signature Σ and each class of models $\mathcal{M} \subseteq |\mathit{Mod}(\Sigma)|$, where $I\!F\mathcal{M}$ denotes the class of all (categorically defined) [7] filtered products over filters in $I\!F$ of families of models in $\mathcal{M}$. Notice that, in a Birkhoff institution, satisfaction is preserved by isomorphisms of models.

Definition 3.2 showed a syntactic counterpart of the general weak amalgamation square concept in Definition 2.1, in the context of first-order logic. A similar notion of weak amalgamation square can be defined for any institution, in particular for Birkhoff institutions.

The following theorem, proved in [10,12], also follows from our Theorem 4.4: [8]

Theorem 6.1 *Given a Birkhoff institution* $(\mathit{Sign}, \mathit{Sen}, \mathit{Mod}, \models, I\!F, B)$ *and a weak amalgamation square as above, any two sets* $\Gamma_1 \subseteq \mathit{Sen}(\Sigma_1)$ *and* $\Gamma_2 \subseteq \mathit{Sen}(\Sigma_2)$ *admit an interpolant provided that:*

- $\mathit{Mod}(\phi_1)$ *preserves products and directed colimits on models.*
- $\mathit{Mod}(\phi_1)$ *lifts* $B_{\Sigma_2}^{-1}$. [9]

Proof.

We take $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ to be $Mod(\Sigma)$, $Mod(\Sigma_1)$, $Mod(\Sigma_2)$, $Mod(\Sigma')$, respectively, and $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$, $\mathcal{V}'$ to be $Mod(\phi_1)$, $Mod(\phi_2)$, $Mod(\phi_1')$, $Mod(\phi_2')$, respectively, all of them considered up to isomorphism of models. (To avoid working with collections of isomorphism classes, one can alternatively take a representative

[7] The meaningful definition of filtered products in an abstract category of models is the one based on directed colimits and direct products.
[8] Note that this theorem does not follow from the results in [33], nor does it imply them.
[9] We have adapted the statement of this theorem to our terminology – thus what in [10] is stated as "ϕ_1 lifts B", here is stated as "$\mathit{Mod}(\phi_1)$ lifts $B_{\Sigma_2}^{-1}$".

for each isomorphism class; thus for example, $Mod(\Sigma)$ would then be a class of non-isomorphic models covering all isomorphism classes, and $Mod(\phi_1)$ would take representatives to corresponding representatives in the image.) Notice that the weak amalgamation property holds for isomorphism classes as well. The relations $B_\Sigma, B_{\Sigma_2}, B_{\Sigma_1}, B_{\Sigma'}$, etc., and the operators of the form $\mathcal{M} \mapsto I\!\!F\mathcal{M}$ are also considered up to isomorphisms – the suitability of these "up-to" relaxations is ensured by the fact that the above relations include the isomorphism relation and by the categorical nature of ultraproducts here. We define $F_\mathcal{A}$ and $F_\mathcal{B}$ as the mappings $\mathcal{M} \mapsto I\!\!F\mathcal{M}$ on $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$, respectively, and $G_\mathcal{A}$ and $G_\mathcal{C}$ as (the operators given by the relations) $(B_\Sigma^{-1})^+$ and $(B_{\Sigma_2}^{-1})^+$ (that is, the transitive closures of B_Σ^{-1} and $B_{\Sigma_2}^{-1}$), respectively. Let us check the hypotheses of Theorem 4.4:

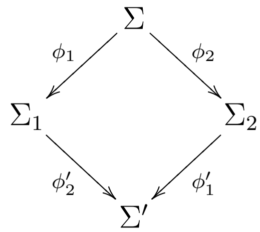
- $G_\mathcal{A}$ and $G_\mathcal{C}$ are closure operators because they are given by reflexive and transitive relations.
- $F_\mathcal{A}; G_\mathcal{A}; F_\mathcal{A} = F_\mathcal{A}; G_\mathcal{A}$ follows from the following:
  · $F_\mathcal{A}; G_\mathcal{A} \sqsubseteq F_\mathcal{A}; G_\mathcal{A}; F_\mathcal{A}$ holds because, since $(\{*\}, \{\{*\}\})$ is in $I\!\!F$, $\mathcal{M} \subseteq I\!\!F\mathcal{M}$ for each $\mathcal{M}$.
  · $F_\mathcal{A}; G_\mathcal{A}; F_\mathcal{A} \sqsubseteq F_\mathcal{A}; G_\mathcal{A}; F_\mathcal{A}; G_\mathcal{A} = F_\mathcal{A}; G_\mathcal{A}$:
    the above inclusion is true because, $(B_\Sigma^{-1})^+$ being reflexive, $G_\mathcal{A}$ is extensive;
    the above equality is true because $F_\mathcal{A}; G_\mathcal{A}$ is an axiomatizability hull operator, thus a closure operator.
- $\mathcal{U}$ preserves fixed points of F because $Mod(\phi_1)$ preserves $I\!\!F$-filtered products.
- $\mathcal{V}$ lifts G is implied, via Proposition 4.5.(4), by $Mod(\phi_2)$ lifting $(B_{\Sigma_2}^{-1}, B_\Sigma^{-1})$.

Now, applying Theorem 4.4, we find a semantic interpolant $\mathcal{K}$ which is a fixed point of both $F_\mathcal{A}$ and $G_\mathcal{A}$, i.e., closed under filtered products and under $(B_\Sigma^{-1})^+$. Moreover, for any class of models $\mathcal{M}$, since $\mathcal{M}^\bullet = B_\Sigma^{-1}(I\!\!F(\mathcal{M}))$, it follows that $\mathcal{M}^\bullet \subseteq \mathcal{B}_\Sigma^{-1}(B_\Sigma^{-1}(I\!\!F(\mathcal{M}))) \subseteq B_\Sigma^{-1}(I\!\!F(B_\Sigma^{-1}(I\!\!F(\mathcal{M})))) = \mathcal{M}^{\bullet\bullet} = \mathcal{M}^\bullet$, and thus $\mathcal{M}^\bullet = B_\Sigma^{-1}(B_\Sigma^{-1}(I\!\!F(\mathcal{M})))$; iterating this, we get $\mathcal{M}^\bullet = (B_\Sigma^{-1})^+(I\!\!F(\mathcal{M}))$, making the semantic interpolant $\mathcal{K}$ a fixed point of $\bullet$, yielding a (syntactic) interpolant $\mathcal{K}^*$. □

6.2 Interpolation from Institutional Quantification

Let us fix an institution. Given a morphism of signatures $\phi : \Sigma \to \Sigma'$, a Σ'-sentence e' and a Σ-model A, we define $A \models (\forall\phi)e'$ as $A' \models e'$ for all Σ'-models A' such that $A'{\restriction}_\phi = A$. An institution is said to *admit universal quantification* [12] provided that for each $\phi : \Sigma \to \Sigma'$ and e as above there exists a sentence e in $Sen(\Sigma)$ semantically equivalent to $(\forall\phi)e'$, in that $A \models e$ iff $A \models (\forall\phi)e'$ for all Σ-models A.

Proposition 6.2 *Consider a weak amalgamation square of signatures as in the diagram*



in an arbitrary institution that admits universal quantification over ϕ_2. Then any two sets of Σ_1- and Σ_2- sentences admit an interpolant.

Proof.

By Theorem 4.4 for $\mathcal{U} = Id$ and $\mathcal{V} = Mod(\phi_2)$, it suffices to show that $Mod(\phi_2)$ lifts the axiomatizability hull operator $\bullet$. Let $\mathcal{M} \subseteq Mod(\Sigma)$. We need to show that $Mod(\phi_2)^{-1}(\mathcal{M}^\bullet) \subseteq Mod(\phi_2)^{-1}(\mathcal{M})^\bullet$. For this, let $A_2 \in Mod(\phi_2)^{-1}(\mathcal{M}^\bullet)$. Then $A_2\restriction_{\phi_2} \in \mathcal{M}^\bullet$. In order to prove $A_2 \in Mod(\phi_2)^{-1}(\mathcal{M})^\bullet$, let e_2 be a Σ-sentence such that $Mod(\phi_2)^{-1}(\mathcal{M}) \models e_2$. Then, for any $B_2 \in Mod(\Sigma_2)$ such that $B_2\restriction_{\phi_2} \in \mathcal{M}$, it is the case that $B_2 \models e_2$; but this precisely means that $\mathcal{M} \models (\forall \phi_2) e_2$, hence $A_2\restriction_{\phi_2} \models (\forall \phi_2) e_2$. From this latter fact and the definition of ϕ_2-quantification, one can deduce $A_2 \models e_2$, which is what we needed. □

Let us apply the above result to obtain interpolation in institutions of second- and higher-order languages. But first, let us briefly describe some unsorted versions of these two institutions.

In the (unsorted) second-order logic, SOL, signatures and models are the same as in unsorted first-order logic, [10] but the first-order sentences are extended by allowing quantification on variables ranging not only over individuals, but also over sets (operations and relations of any arity). Satisfaction is the usual second-order satisfaction. The behavior of the functors *Sign* and *Mod* on signature morphisms is the natural one.

To define higher-order logic, HOL, let b be a fixed symbol that will stand for the *basic type*. The set T of types is defined recursively by the following rules:

(1) $b \in T$.
(2) if $t_1, \ldots, t_n \in T$, then $t_1 * \ldots * t_n \in T$.
(3) if $t \in T$, then $Pt \in T$, where one should regard P as a type constructor.

A *higher-order signature* is a T-indexed set $\Sigma = (\Sigma_t)_{t \in T}$, the elements of Σ_t being called *constants of type* t. A morphism between Σ and Σ' is a T-indexed mapping $\phi = (\phi_t)_{t \in T}$, where $\phi_t : \Sigma_t \to \Sigma'_t$ for all $t \in T$. To each set

[10] Since signatures have only one sort, we can omit it and regard the signatures as pairs (F, P).

D, one naturally associates the T-indexed set $(D_t)_{t\in T}$ as follows: $D_b = D$, $D_{t_1*\ldots*t_n} = D_{t_1} \times \ldots \times D_{t_n}$, $D_{Pt} = \mathcal{P}(D_t)$. A Σ-model is a structure of the form $(A, (A_t(c))_{t\in T, c\in\Sigma_t})$, where $A_t(c) \in A_t$ for each $t \in T$ and $c \in \Sigma_t$. The Σ-terms are constants or variables of any type. The Σ-atoms have the form $u(v_1, \ldots, v_n)$ with the type of u being $P(t_1 * \ldots * t_n)$ where each t_i is the type of v_i. The Σ-sentences are built from atoms by means of the usual connectives and quantifiers. The satisfaction relation, as well as the mappings on sentences and models associated with signature morphisms are the natural ones.

Interpolation for right-finite weak amalgamation squares in SOL and HOL logics follow as a corollary of Proposition 6.2:

Proposition 6.3 *Consider a SOL or HOL weak amalgamation square of signatures as in the diagram above such that the signatures Σ and Σ_2 are finite.*[11] *Then any two sets of Σ_1- and Σ_2- sentences admit an interpolant.*

Proof.

All we need to notice is that SOL and HOL admit universal quantification over any morphism $\phi_2 : \Sigma \to \Sigma_2$ between finite signatures – for any sentence e_2 in $Sen(\Sigma_2)$, $(\forall\phi_2)e_2$ is semantically equivalent:

- In SOL, to $\forall F_2 \setminus \phi^{op}(F).\forall P_2 \setminus \phi^{rel}(P).\, \big(\bigwedge_{f,g\in F, \phi^{op}(f)=\phi^{op}(g)} Eql(f,g) \wedge \bigwedge_{p,r\in P, \phi^{rel}(p)=\phi^{rel}(r)} Eql(p,r)\big) \Rightarrow e_2$, where $Eql(f,g)$ and $Eql(p,r)$ are the usual syntactic sugar saying that two second-order items (functions or relations) are equal – in the above sentence, we have used function and relation symbols in $\Sigma_2 \setminus \Sigma$ as second-order variables and quantified over them in Σ.
- In HOL, to $\forall\Sigma_2\setminus\Sigma.\, \big(\bigwedge_{t\in T,\, c,d\in\Sigma_t,\, \phi^t(c)=\phi^t(d)} Eql(c,d)\big) \Rightarrow e_2$, applying a similar technique as for SOL.

□

In the above proof, we used that Σ and Σ_2 were finite (so that the constructed sentence was finitary) – this hypothesis could actually be relaxed to requiring that Σ_2 adds only finitely many items to the image $\phi_2(\Sigma)$ and that ϕ_2 identifies only finitely many pairs of items from Σ. Also, the proof makes crucial use of the fact that equality is definable in these logics – this means that we either work with standard models as we did above, or we work with Henkin models and take the equality symbol as a primitive, interpreted in models as actual equality.

[11] In HOL, a signature being finite means that it has only a finite number of constants.

7 Applications to Formal Specification

Craig interpolation is a an important/desired property in many areas. Next we consider some applications of our interpolation results to formal specification and module algebra.

In formalisms for modularization [3,15,37], modules are built by composing other modules via specific operations. One typically starts with *flat* (or *basic*) modules, which are pairs (Σ, Γ) comprising a signature Σ and a set of Σ-sentences Γ. According to [3], one of the most natural semantics of modules, also called *flat semantics*, is given by their corresponding theories; for example, the semantics of a basic module (Σ, Γ) is the theory $(\Sigma, \Gamma^\bullet)$. Diverse operations are used to build up structured theories, among which the *export* (or *information hiding*) and *combination* (or *sum*) operators [3] (or [15]), $\Box$ and $+$. $\Box$ restricts the interface of the theory (Σ, Γ) to common symbols of Σ' and Σ, while $+$ just puts together two theories in their union signature. Formally, for each signature Σ' and theory (Σ, Γ), let $\Sigma' \Box (\Sigma, \Gamma)$ be $(\Sigma' \cap \Sigma, \iota^{-1}(\Gamma))$, where $\iota : \Sigma' \cap \Sigma \hookrightarrow \Sigma$; and for theories (Σ_1, Γ_1) and (Σ_2, Γ_2), let $(\Sigma_1, \Gamma_1) + (\Sigma_2, \Gamma_2)$ be $(\Sigma_1 \cup \Sigma_2, (\Gamma_1 \cup \Gamma_2)^\bullet)$. A desirable property of specification frameworks is the following *restricted distributivity law*:

$$\Sigma' \Box ((\Sigma_1, \Gamma_1) + (\Sigma_2, \emptyset^\bullet)) = (\Sigma' \Box (\Sigma_1, \Gamma_1)) + (\Sigma' \Box (\Sigma_2, \emptyset^\bullet))$$

As discussed in [3,15], full distributivity does not typically hold. It is shown in [3] that, in first-order logic, restricted distributivity is implied by interpolation. Their proof is rather logic-independent, so it works for any logic that has first-order signatures and satisfies interpolation. In particular, it works for all the sub-logics of (finitary or infinitary) first-order logic appearing in Table 1. Thus our interpolation results show that the restricted distributivity law holds in module algebra developed within many logical frameworks intermediate between full first-order logic and equational logic.

Another application to formal specifications relies on the fact that interpolation entails a *compositional behavior* of the semantics of structured specifications, by ensuring that the two alternative semantics, the flat and the structured ones, coincide. There are good reasons to *not* always consider the flat semantics of module expressions, but rather to *keep the structure* of modules [37,5,22]. In the case of hiding, $\Sigma' \Box (\Sigma, \Gamma)$ provides more information than $(\Sigma', \Gamma^\bullet \cap Sen(\Sigma'))$: (1) Γ might be finite, showing that $\Gamma^\bullet$, maybe unlike $\Gamma^\bullet \cap Sen(\Sigma')$, is finitely presented; (2) while the theory of all Σ'-reducts of (Σ, Γ) (i.e., all visible parts of the possible implementations of the theory) is indeed $\Gamma^\bullet \cap Sen(\Sigma')$, usually not any model of $\Gamma^\bullet \cap Sen(\Sigma')$ is a Σ-reduct of a model of (Σ, Γ); hence the theory does not describe precisely the intended semantics on classes of models.

To understand the role played by interpolation, consider the situation when a module $\Sigma'\Box(\Sigma,\Gamma)$ is imported and its interface (Σ') is renamed via a signature morphism $j:\Sigma'\to\Sigma''$ in the importing context. The flat semantics of the renamed module is $(\Sigma'', j(\Gamma^\bullet\cap \mathit{Sen}(\Sigma'))^\bullet)$. On the other hand, the renamed module itself might be regarded constructively as an information hiding module whose interface is Σ'' and whose base module is a consistent renaming of (Σ,Γ). This is achieved by taking the pushout $(\Sigma''\hookrightarrow\Sigma_0, j_0:\Sigma\to\Sigma_0)$ of $(\Sigma'\hookrightarrow\Sigma, j:\Sigma'\to\Sigma'')$, yielding the new module $\Sigma''\Box(\Sigma_0, j_0(\Gamma))$. One can show *using interpolation* that the modular and the flat semantics are equivalent, that is, $j(\Gamma^\bullet\cap \mathit{Sen}(\Sigma'))^\bullet = j_0(\Gamma)^\bullet\cap \mathit{Sen}(\Sigma'')$. This desirable semantical equivalence is shown by our results to hold for several first-order sub-logics. More precisely, lines 3,5,15 in Table 1 show that the framework may be restricted to positive-, universal-, or [universal quantification of atom disjunction]- logics. Moreover, line 21 shows the same thing for the [universal quantification of possibly infinite atom disjunction]-logic. According to these results, the renaming morphism j can be allowed to be injective *on sorts* in the case of positive logic and *any* morphism in the other three cases. Note that lines 2,4,14,20 list results complementary to the above, and generalize those in [10]. These latter results relax the requirements not on the renaming morphism, but on the *hiding morphism* (allowing one to replace the inclusion $\Sigma'\hookrightarrow\Sigma$ with an arbitrary signature morphism).

Within a specification framework, one should not commit to a particular kind of first-order sub-logic, but rather use the available power of expression on a by-need basis, keeping flexible the border between expressive power and effective/efficient decision or computation. The issue of coexistence of different logical systems brings up a third application of our results. The various logical systems that one would like to use should not be simply “swallowed” by a richer universal logic that encompasses them all, but rather integrated using logic translations. This methodology, which is the meta-logical counterpart of keeping structured (i.e., unflattened) the specifications themselves, is followed for instance in CafeOBJ [13,14]. The underlying logical structure of this system can be formalized as a *Grothendieck institution* [9], which provides a means of building specifications inside the minimal needed logical system. The framework is initially presented as an *indexed institution*, i.e., a family of logical systems with translations between them, and then flattened by a Grothendieck construction.

Lifting interpolation from the component institutions to the Grothendieck institution was studied in [11]; a criterion is given there for lifting interpolation, consisting mainly of three conditions: (1) that the component institutions have interpolation (for some designated pushouts of signatures); (2) that the involved institution comorphisms have interpolation; (3) that each pullback in the index category yields an interpolating square of comorphisms. We give just one example showing that, via the above conditions, some of our interpolation results can be used for putting together in a consistent way two very interesting logical systems: (finitary) first-order logic ($\mathcal{FO}$) and the logic of universally quantified possibly infinite disjunctions of atoms ($\forall\vee_\infty$). While the former is a well-established logic, the latter has the ability of expressing some important properties, not expressible in the former, such as *accessibility* of models, e.g., $(\forall x)(x = 0 \vee x = s(0) \vee x = s(s(0)) \vee \ldots)$ for natural numbers. If one combines the expressive power of these two logics, initiality conditions are also available, e.g., the above accessibility condition ("no junk") can be complemented with the "no confusion" statement $\neg \bigvee_{i,j\in\mathbb{N},i<j} s^i(0) = s^j(0)$. Since the two logical systems have the same signatures, condition (2) above is trivially satisfied. Moreover, our results stated in lines 1 and 21 of Table 1 ensure condition (1) for some very wide class of signature pushouts. Finally, condition (3) is fulfilled by the result in line 22, which states that formulae from the two logics have interpolants *in their intersection logic*, that of universally quantified (finite) conjunctions of atoms.

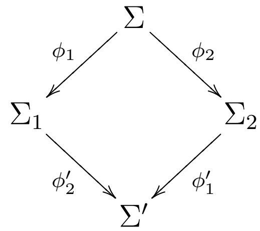
8 Craig-Robinson Interpolation

Throughout this paper, we used the term "interpolation" to mean "Craig interpolation" (abbreviated CI). A related stronger property is *Craig-Robinson interpolation (CRI)*. Some researchers [24,16] have argued that CRI, and not just CI, is desirable in algebraic specification frameworks. First-order logic has CRI, and so does any logic which has CI, is compact and has implications [12]. However, this is not the case with most of the important fragments of first-order logic, like Horn logic; in fact, most of the logics in Table 1 do not have CRI (unless one takes some harsh restrictions on the signature morphisms).

We can apply our semantic technique, to some extent, to obtain a general CRI theorem, too. A sketch of a semantic approach to CRI is discussed next (we omit the proofs, because they are quite similar to those for CI).

Definition 8.1 *Assume a commutative square of signature morphisms (see diagram below) and three sets of sentences* $\Gamma_1 \subseteq Sen(\Sigma_1)$*, and* $\Delta, \Gamma_2 \subseteq Sen(\Sigma_2)$ *such that* $\phi_2'(\Gamma_1) \cup \phi_1'(\Delta) \models_{\Sigma'} \phi_1'(\Gamma_2)$*. A* ***Craig-Robinson interpolant (CR interpolant)*** *for* Γ_1, Γ_2 *through* Δ *is a set* $\Gamma \subseteq Sen(\Sigma)$ *such that* $\Gamma_1 \models_{\Sigma_1} \phi_1(\Gamma)$

and $\phi_2(\Gamma) \cup \Delta \models_{\Sigma_2} \Gamma_2$.

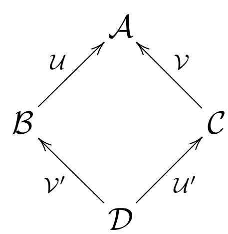


Just like CI, CRI can be regarded semantically, using the equivalences below:

- $\phi'_2(\Gamma_1) \cup \phi'_1(\Delta) \models \phi'_1(\Gamma_2)$ iff $(\phi'_2(\Gamma_1) \cup \phi'_1(\Delta))^* \subseteq \phi'_1(\Gamma_2)^*$ iff $\phi'_2(\Gamma_1)^* \cap \phi'_1(\Delta)^* \subseteq \phi'_1(\Gamma_2)^*$ iff $Mod(\phi'_2)^{-1}(\Gamma_1^*) \cap Mod(\phi'_1)^{-1}(\Delta^*) \subseteq Mod(\phi'_1)^{-1}(\Gamma_2^*)$.
- $\Gamma_1 \models \phi_1(\Gamma)$ iff $\Gamma_1^* \subseteq \phi_1(\Gamma)^*$ iff $\Gamma_1^* \subseteq Mod(\phi_1)^{-1}(\Gamma^*)$.
- $\phi_2(\Gamma) \cup \Delta \models \Gamma_2$ iff $(\phi_2(\Gamma) \cup \Delta)^* \subseteq \Gamma_2^*$ iff $\phi_2(\Gamma)^* \cap \Delta^* \subseteq \Gamma_2^*$ iff $Mod(\phi_2)^{-1}(\Gamma^*) \cap \Delta^* \subseteq \Gamma_2^*$ iff $Mod(\phi_2)^{-1}(\Gamma^*) \subseteq \Gamma_2^* \cup \overline{\Delta^*}$.

Therefore, one can define *semantic CR interpolants* as follows:

Definition 8.2 *Consider the commutative diagram*



together with some $\mathcal{M} \in \mathcal{P}(\mathcal{B})$ *and* $\mathcal{N}, \mathcal{R} \in \mathcal{P}(\mathcal{C})$ *such that* $\mathcal{V}'^{-1}(\mathcal{M}) \cap \mathcal{U}'^{-1}(\mathcal{R}) \subseteq \mathcal{U}'^{-1}(\mathcal{N})$. *We say that* $\mathcal{K} \in \mathcal{P}(\mathcal{A})$ *is a* ***semantic CR interpolant*** *of* $\mathcal{M}$ *and* $\mathcal{N}$ *through* $\mathcal{R}$ *iff* $\mathcal{M} \subseteq \mathcal{U}^{-1}(\mathcal{K})$ *and* $\mathcal{V}^{-1}(\mathcal{K}) \cap \mathcal{R} \subseteq \mathcal{N}$.

If we take $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ to be $Mod(\Sigma)$, $Mod(\Sigma_1)$, $Mod(\Sigma_2)$, $Mod(\Sigma')$ and $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$, $\mathcal{V}'$ to be $Mod(\phi_1)$, $Mod(\phi_2)$, $Mod(\phi'_1)$, $Mod(\phi'_2)$, respectively, we obtain the concrete first-order case.

Let $\mathcal{CRI}(\mathcal{M}, \mathcal{N}, \mathcal{R})$ denote the collection of all CR semantic interpolants of $\mathcal{M}$ and $\mathcal{N}$ through $\mathcal{R}$. The following two results characterize the semantic CR interpolants and give criteria for their existence (the second taking fixed points into account).

Proposition 8.3 *Under the hypothesis of Definition 8.2:*

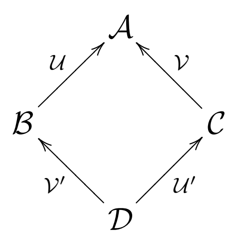
(1) $\mathcal{CRI}(\mathcal{M}, \mathcal{N}, \mathcal{R}) = [\mathcal{U}(\mathcal{M}), \overline{\mathcal{V}(\overline{\mathcal{N}} \cap \mathcal{R})}]$.
(2) If the square is a weak amalgamation square then $\mathcal{CRI}(\mathcal{M}, \mathcal{N}, \mathcal{R}) \neq \emptyset$.

Theorem 8.4 *Consider a weak amalgamation square as in the diagram below*

	Γ_1 Type	Γ_2, Δ Type	Γ Type	$Mod(\phi_1)$ preserves	$Mod(\phi_2)$ lifts	ϕ_1	ϕ_2
1	$\mathcal{FO}$	$\mathcal{FO}$	$\mathcal{FO}$	Up	Ur	any	(IS)
2	$\mathcal{P}os$	$\mathcal{FO}$	$\mathcal{P}os$	$Up\,;H$	Ur	(IS), (IR), (RS)	(IS)
3	$\forall$	$\mathcal{FO}$	$\forall$	$Up\,;S$	Id	(IS), (RS)	any
4	$\exists$	$\mathcal{FO}$	$\exists$	$Up\,;Ext$	Ur	(I)	(IS)
5	$\mathcal{UH}$	$\mathcal{FO}$	$\mathcal{UH}$	$Fp\,;S$	Id	(IS), (RS)	any
6	$\mathcal{UA}$	$\mathcal{FO}$	$\mathcal{UA}$	$P\,;S\,;H$	Id	(IS), (IR), (RS)	any
7	$\forall\vee$	$\mathcal{FO}$	$\forall\vee$	$Up\,;S\,;Hs$	Id	(IS), (RS)	any
8	$\Box$	$\mathcal{FO}$	$\Box$	$Up\,;Hs$	Id	(IS), (RS)	any
9	$\mathcal{UH}_\infty$	$\mathcal{FO}_\infty$	$\mathcal{UH}_\infty$	$P\,;S$	Id	(IS), (RS)	any
10	$\forall\vee_\infty$	$\mathcal{FO}_\infty$	$\forall\vee_\infty$	$S\,;Hs$	Id	(IS), (RS)	any

Table 2
Craig-Robinson interpolation results for first-order sub-logics.

and two pairs of operators $F = (F_{\mathcal{B}}, F_{\mathcal{A}})$ and $G = (G_{\mathcal{C}}, G_{\mathcal{A}})$ such that:



(1) $F_{\mathcal{A}}\,;G_{\mathcal{A}}\,;F_{\mathcal{A}} = F_{\mathcal{A}}\,;G_{\mathcal{A}}$.
(2) $G_{\mathcal{C}}$ and $G_{\mathcal{A}}$ are closure operators.
(3) $\mathcal{U}$ preserves fixed points of F.
(4) $\mathcal{V}$ lifts G.

Then for each $\mathcal{M} \in Fixed(F_{\mathcal{B}})$ and $\mathcal{N} \cup \overline{\mathcal{R}} \in Fixed(G_{\mathcal{C}})$ such that $\mathcal{V}'^{-1}(\mathcal{M}) \subseteq \mathcal{U}'^{-1}(\mathcal{N})$, $\mathcal{M}$ and $\mathcal{N}$ have through $\mathcal{R}$ a semantic interpolant $\mathcal{K}$ in $Fixed(F_{\mathcal{A}}) \cap Fixed(G_{\mathcal{A}})$.

Note that Theorem 8.4 is almost identical with its CI counterpart, Theorem 4.4 – the hypotheses (1)-(4) are the same and the conclusion has the same format. The only difference is that one requires that $\mathcal{N} \cup \overline{\mathcal{R}}$, rather than just $\mathcal{N}$, to be a fixed point of $G_{\mathcal{C}}$. That our semantic approach works for CRI as well is very fortunate; however, as we see below, the application to concrete cases is much restricted compared to CI due to the resulted requirement that $\mathcal{N} \cup \overline{\mathcal{R}}$ be axiomatizable.

Table 2 states CR interpolation results for various first-order sub-logics. It should be read as: given a weak amalgamation square of signatures as in Definition 8.1, if $\Gamma_1 \subseteq Sen(\Sigma_1)$, $\Gamma_2 \subseteq Sen(\Sigma_2)$, and $\Delta \subseteq Sen(\Sigma_2)$ are sets of sentences of the indicated types such that $\phi_2'(\Gamma_1) \cup \phi_1'(\Delta) \models \phi_1'(\Gamma_2)$, then they have a CR interpolant Γ of the indicated type; the semantic conditions under which this situation holds are again given in the $Mod(\phi_1)$- and $Mod(\phi_2)$-columns of the table, with the meaning that $Mod(\phi_1)$ *preserves* fixed points of the indicated operator and $Mod(\phi_2)$ *lifts* the indicated operator.

Theorem 8.5 *The results stated in Table 2 hold, i.e., in each of the 10 cases, if ϕ_1 and ϕ_2 satisfy the indicated properties, Γ_1, Γ_2 and Δ have the indicated types and $\phi_2'(\Gamma_1) \cup \phi_1'(\Delta) \models \phi_1'(\Gamma_2)$, then there exists a CR interpolant Γ of the indicated type.*

As an interesting (but admittedly not very significant for algebraic specification) consequence of the above results, it follows from line 5 of the table that Horn logic without operation symbols has Craig-Robinson interpolation.

Notice that the CRI results in Table 2 correspond to the CI results in Table 1 for which the sentences on the righthand side are *arbitrary* finitary or infinitary first-order sentences (that is, $\mathcal{FO}$ and $\mathcal{FO}_\infty$), because these are the only cases in Table 1 where the axiomatizable classes of models are closed under complement and union, as required to deal with the expression $\mathcal{N} \cup \overline{\mathcal{R}}$ from Theorem 8.4.

9 Related Work and Concluding Remarks

The idea of using axiomatizability properties for proving Craig interpolation first appeared, up to our knowledge, in [35] in the case of many-sorted equational logic. Then [32] generalized this to an arbitrary pullback of categories, by considering some Birkhoff-like operators on those categories, with results applicable to different versions of equational logic.

An institution-independent relationship between Birkhoff-like axiomatizability and Craig interpolation was depicted in [10], using the concept of *Birkhoff institution*. Moreover, [17] studies insitution-independent Robinson consistency, which is equivalent to Craig interpolation for any compact logic and admitting finite conjunctions and negations (in particular, for FOL, but *not* for any of its sub-logics discussed in our paper). If we disregard the combination of logics and flatten to the least logic, the results in lines 2,4,14,20 of Table 1 can be also found in [10], and the result in line 1 of Table 1 can be also found in [6] and [17]. Our Theorem 4.4 generalizes the previous "semantic" results, bringing the technique of semantic interpolation, we might say, up to its limit.

The merit of Theorem 4.4 is that it provides general conditions under which a semantic interpolant has a syntactic counterpart (i.e., it is axiomatizable). This theorem solves only half of the interpolation problem; concrete lifting and preserving conditions, as well as certain inclusions between operators, still have to be proved. Thus, in this paper, we provide a general methodology for proving interpolation results. Following this methodology, we worked out many concrete examples.

The list of first-order sub-logics that fit our framework is open to any other suitably axiomatizable logics; and so are the possible combinations between these logics, which might guarantee interpolants even simpler than the types of formulae of *both* logics, as shown by some of our results. Regarding our combined interpolation results, it is worth pointing out that they are not overlapped with, but rather complementary to, the ones in [11] for Grothendieck institutions. There, some combined interpolation properties are previously assumed, in order to ensure interpolation in the resulted larger logical system. As for Craig-Robinson interpolation, the only general treatment of this property that we are aware of is the monograph [12], which relates Craig and Craig-Robinson interpolations in an arbitrary institution which is compact and admits negation.

An interesting fact to investigate would be to which extent can syntactically-obtained interpolation results compete with our semantic results. While it is true that the syntactic proofs are sometimes more constructive, they do not seem to provide information on the type of the interpolant comparable to what we gave here. In particular, since the diverse Gentzen systems for first-order logic with equality have only partial cut elimination [18], an appeal to the non-equality version of the language, by adding appropriate axioms for equality in the theory, is needed; moreover, dealing with function symbols requires a further appeal to an encoding of functions as relations, again with the cost of adding some axioms. All these transformations make syntactic proofs rather indirect and obliterating, and sometimes place the interpolant way outside the given subtheory – this is probably the reason why an interpolation theorem for equational logic was not known until a separate, equational-logic-specific proof was given in [36]. Comparing and paralleling (present or future) semantic and syntactic proofs seems fruitful for deepening our understanding of Craig interpolation, such a purely syntactic and yet surprisingly semantic property for logical systems.

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References

[1] H. Andréka and I. Németi. A general axiomatizability theorem formulated in terms of cone-injective subcategories. In *Universal Algebra (Proc. Coll. Esztergom 1977) Colloq. Math. Soc. J. Bolyai*, volume 29, pages 13–35. North-Holland, Amsterdam, 1981.

[2] J. Barwise and J. Feferman. *Model-Theoretic Logics*. Springer, 1985.

[3] J. Bergstra, J. Heering, and P. Klint. Module algebra. *Journal of the Association for Computing Machinery*, 37(2):335–372, 1990.

[4] T. Borzyszkowski. Generalized interpolation in CASL. *Information Processing Letters*, 76(1-2):19–24, 2000.

[5] T. Borzyszkowski. Logical systems for structured specifications. *Theoretical Computer Science*, 286(2):197–245, 2002.

[6] T. Borzyszkowski. Generalized interpolation in first order logic. *Fundamenta Informaticae*, 66(3):199–219, 2005.

[7] C. C. Chang and H. J. Keisler. *Model Theory*. North Holland, Amsterdam, 1973.

[8] W. Craig. Linear reasoning. A new form of the Herbrand-Gentzen Theorem. *Journal of Symbolic Logic*, 22:250–268, 1957.

[9] R. Diaconescu. Grothendieck institutions. *Applied Categorical Structures*, 10(4):383–402, 2002.

[10] R. Diaconescu. An institution-independent proof of Craig interpolation theorem. *Studia Logica*, 77(1):59–79, 2004.

[11] R. Diaconescu. Interpolation in Grothendieck institutions. *Theoretical Computer Science*, 311:439–461, 2004.

[12] R. Diaconescu. *Institution-independent Model Theory*. Birkhauser, 2008.

[13] R. Diaconescu and K. Futatsugi. *CafeOBJ Report*. World Scientific, 1998. AMAST Series in Computing, volume 6.

[14] R. Diaconescu and K. Futatsugi. Logical foundations of CafeOBJ. *Theoretical Computer Science*, 285:289–318, 2002.

[15] R. Diaconescu, J. Goguen, and P. Stefaneas. Logical support for modularization. In G. Huet and G. Plotkin, editors, *Logical Environments*, pages 83–130. Cambridge, 1993.

[16] T. Dimitrakos and T. Maibaum. On a generalized modularization theorem. *Information Processing Letters*, 74(1-2):65–71, 2000.

[17] D. Găină and A. Popescu. An institution-independent proof of the Robinson consistency theorem. *Studia Logica*, 85(1):41–73, 2007.

[18] J. H. Gallier. *Logic for computer science. Foundations of automatic theorem proving.* Harper & Row, 1986.

[19] J. Goguen. Types as theories. In *Topology and Category Theory in Computer Science*, pages 357–390. Oxford, 1991.

[20] J. Goguen and R. Burstall. Introducing institutions. In E. Clark, editor, *Proceedings, Logics of Programming Workshop, Lecture Notes in Computer Science*, volume 164, pages 221–256. Springer-Verlag, 1984.

[21] J. Goguen and R. Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, January 1992.

[22] J. Goguen and G. Roşu. Composing hidden information modules over inclusive institutions. In *From Object Orientation to Formal Methods: Dedicated to the memory of Ole-Johan Dahl*, volume 2635 of *Lecture Notes in Computer Science*, pages 96–123. Springer, 2004.

[23] D. G. J. Bicarregui, T. Dimitrakos and T. Maibaum. Interpolation in practical formal development. *Logic Journal of the Interest Group in Pure and Applied Logic*, 9(1):231–243, 2001.

[24] T. S. E. Maibaum and M. R. Sadler. Axiomatizing specification theory. In H.-J. Kreowski, editor, *ADT*, volume 116 of *Informatik-Fachberichte*, pages 171–177. Springer, 1984.

[25] J. Meseguer. Membership algebra as a logical framework for equational specification. In *Proceedings of WADT'97*, volume 1376 of *Lecture Notes in Computer Science*, pages 18–61. Springer, 1998.

[26] J. D. Monk. *Mathematical Logic.* Springer-Verlag, 1976.

[27] T. Mossakowski, J. Goguen, R. Diaconescu, and A. Tarlecki. What is a logic? In J.-Y. Beziau, editor, *Logica Universalis*, pages 113–133. Birkhauser, 2005.

[28] G. Nelson and D. C. Oppen. Simplification by cooperating decision procedures. *ACM Transactions on Programming Languages and Systems*, 1(2):245–257, 1979.

[29] I. Németi and I. Sain. Cone-implicational subcategories and some Birkhoff-type theorems. In *Universal Algebra (Proc. Coll. Esztergom 1977) Colloq. Math. Soc. J. Bolyai*, volume 29, pages 535–578. North-Holland, Amsterdam, 1981.

[30] D. C. Oppen. Complexity, convexity and combinations of theories. *Theoretical Computer Science*, 12:291–302, 1980.

[31] A. Popescu, T. Serbanuta, and G. Rosu. A semantic approach to interpolation. In L. Aceto and A. Ingólfsdóttir, editors, *FoSSaCS*, volume 3921 of *Lecture Notes in Computer Science*, pages 307–321. Springer, 2006. also appeared as Technical Report UIUCDCS-R-2005-2643, May 2005.

[32] G. Roşu and J. Goguen. On equational Craig interpolation. *Journal of Universal Computer Science*, 6:194–200, 2000.

[33] G. Roşu and J. Goguen. On equational Craig interpolation. *Journal of Universal Computer Science*, 6(1):194–200, 2000.

[34] P. H. Rodenburg. Interpolation in conditional equational logic. *Fundamenta Informaticae*, 15(1):80–85, 1991.

[35] P. H. Rodenburg. A simple algebraic proof of the equational interpolation theorem. *Algebra Universalis*, 28:48–51, 1991.

[36] P. H. Rodenburg and R. van Glabbeek. An interpolation theorem in equational logic. Technical Report CS-R8838, CWI, 1988.

[37] D. Sannella and A. Tarlecki. Specifications in an arbitrary institution. *Information and Control*, 76:165–210, 1988.

APPENDIX - Proofs for some of the facts in the paper

Proposition 4.5 *Consider two classes $\mathcal{C}$ and $\mathcal{D}$, a mapping $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ and a pair of relations $R = (R_{\mathcal{C}}, R_{\mathcal{D}})$, with $R_{\mathcal{C}} \subseteq \mathcal{C} \times \mathcal{C}$ and $R_{\mathcal{D}} \subseteq \mathcal{D} \times \mathcal{D}$.*[12] *Then the following hold:*

(1) $\mathcal{U}$ lifts R if and only if for any elements $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $dR_{\mathcal{D}}\mathcal{U}(c)$, there exists $c' \in \mathcal{C}$ such that $\mathcal{U}(c') = d$ and $c'R_{\mathcal{C}}c$.
(2) Suppose $R_{\mathcal{C}}$ is reflexive and transitive. Then $\mathcal{U}$ preserves fixed points of R if and only if for all elements $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $\mathcal{U}(c)R_{\mathcal{D}}d$, there exists $c' \in \mathcal{C}$ such that $\mathcal{U}(c') = d$ and $cR_{\mathcal{C}}c'$.
(3) Suppose $R_{\mathcal{C}}$ is reflexive and transitive. Then $\mathcal{U}$ preserves fixed points of R if and only if $\mathcal{U}$ lifts $(R_{\mathcal{C}}^{-1}, R_{\mathcal{D}}^{-1})$.
(4) Let R^+ be the pair $(R_{\mathcal{C}}^+, R_{\mathcal{D}}^+)$, where $R_{\mathcal{C}}^+$ and $R_{\mathcal{D}}^+$ are the transitive closures of $R_{\mathcal{C}}$ and $R_{\mathcal{D}}$ respectively. Then $\mathcal{U}$ lifts R^+ if $\mathcal{U}$ lifts R.

Proof.

1. Assume that $\mathcal{U}$ lifts R and let $c \in \mathcal{C}$ and $d \in \mathcal{D}$ be two elements such that $dR_{\mathcal{D}}\mathcal{U}(c)$. Then $\mathcal{U}(c) \in R_{\mathcal{D}}(\{d\})$, thus $c \in R_{\mathcal{C}}(\mathcal{U}^{-1}(\{d\}))$, i.e., there exists $c' \in \mathcal{C}$ such that $\mathcal{U}(c') \in \{d\}$ and $c \in R_{\mathcal{C}}(\{c'\})$. But the latter just mean $\mathcal{U}(c') = d$ and $c'R_{\mathcal{C}}c$. Conversely, let $\mathcal{D}' \in \mathcal{P}(\mathcal{D})$ and $c \in \mathcal{C}$ such that $\mathcal{U}(c) \in R_{\mathcal{D}}(\mathcal{D}')$. Then there exists $d \in \mathcal{D}'$ such that $dR_{\mathcal{D}}\mathcal{U}(c)$. Thus, there exists $c' \in \mathcal{C}$ such that $\mathcal{U}(c') = d$ and $c'R_{\mathcal{C}}c$. But this implies $c' \in \mathcal{U}^{-1}(\mathcal{D}')$, and furthermore $c \in R_{\mathcal{C}}(\mathcal{U}^{-1}(\mathcal{D}'))$.

2. Suppose $\mathcal{U}$ preserves fixed points of R and let $c \in \mathcal{C}$ and $d \in \mathcal{D}$ be two elements such that $\mathcal{U}(c)R_{\mathcal{D}}d$. Since $R_{\mathcal{C}}$ is reflexive and transitive we have that

[12] Recall that $R_{\mathcal{C}}$ and $R_{\mathcal{D}}$ also denote the induced operators.

$R_{\mathcal{C}}(\{c\})$ is a fixed point of $R_{\mathcal{C}}$, so $\mathcal{U}(R_{\mathcal{C}}(\{c\}))$ must be a fixed point of $R_{\mathcal{D}}$. Since $\mathcal{U}(c) \in \mathcal{U}(R_{\mathcal{C}}(\{c\}))$ and $\mathcal{U}(c) R_{\mathcal{D}} d$, it follows that $d \in \mathcal{U}(R_{\mathcal{C}}(\{c\}))$, whence there exist $c' \in R_{\mathcal{C}}(\{c\})$ such that $\mathcal{U}(c') = d$. But $c' \in R_{\mathcal{C}}(\{c\})$ means exactly that $c R_{\mathcal{C}} c'$. Conversely, let $\mathcal{C}'$ be a fixed point of $R_{\mathcal{C}}$. We want to show that $\mathcal{U}(\mathcal{C}')$ is a fixed point of $R_{\mathcal{D}}$. Let $d \in R_{\mathcal{D}}(\mathcal{U}(\mathcal{C}'))$. There exists $c \in \mathcal{C}'$ such that $\mathcal{U}(c) R_{\mathcal{D}} d$ whence there exists $c' \in \mathcal{C}$ such that $\mathcal{U}(c') = d$ and $c R_{\mathcal{C}} c'$. Since $c \in \mathcal{C}'$ and $\mathcal{C}'$ is a fixed point of $R_{\mathcal{C}}$, it follows that $c' \in \mathcal{C}'$, whence $d \in \mathcal{U}(\mathcal{C}')$.

3. Obvious from points 1 and 2.

4. Assume $\mathcal{U}$ lifts R and let $c \in \mathcal{C}$ and $d \in \mathcal{D}$ such that $d R^{+}_{\mathcal{D}} \mathcal{U}(c)$. Then there exist $d_1, \ldots, d_k \in \mathcal{D}$ such that $d_1 = d$, $d_k = \mathcal{U}(c)$, and $d_i R_{\mathcal{D}} d_{i+1}$ for each $i \in \{1, \ldots, k-1\}$. Iterating the R-lifting property, we obtain, successively, elements $c_k, \ldots, c_1 \in \mathcal{C}$ such that $c_k = c$ and $c_i R_{\mathcal{C}} c_{i+1}$ and $\mathcal{U}(c_i) = d_i$ for each $i \in \{1, \ldots, k-1\}$. We thus found $c_1 \in \mathcal{C}$ such that $\mathcal{U}(c_1) = d_1 = d$ and $c_1 R^{*}_{\mathcal{C}} c$, as desired. □

Proposition 4.6 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be classes and consider the following diagram:*

$$\mathcal{P}(\mathcal{A}) \xrightarrow{V} \mathcal{P}(\mathcal{B}) \underset{U'}{\overset{U}{\rightrightarrows}} \mathcal{P}(\mathcal{C}) \xrightarrow{V'} \mathcal{P}(\mathcal{D})$$

such that $U \sqsubseteq U'$. Then:

(1) $V \,; U \sqsubseteq V \,; U'$.
(2) V' monotone implies that $U \,; V' \sqsubseteq U' \,; V'$.

Proof.

1. Clearly $U(V(\mathcal{A}')) \subseteq U'(V(\mathcal{A}'))$ for all $\mathcal{A}' \subseteq \mathcal{A}$.

2. Let $\mathcal{B}' \subseteq \mathcal{B}$. Then $U(\mathcal{B}') \subseteq U'(\mathcal{B}')$, hence $V'(U(\mathcal{B}')) \subseteq V'(U'(\mathcal{B}'))$ by the monotonicity of V'. □

Proposition 4.7 *Let F and G be operators on the same class $\mathcal{D}$ such that F is a closure operator. The following hold:*

(1) If $G \,; F \sqsubseteq F \,; G$ then $F \,; G \,; F = F \,; G$.
(2) If G is also a closure operator, then $F \,; G$ is a closure operator iff $G \,; F \sqsubseteq F \,; G$.

Proof.

1. We use Proposition 4.6, together with the idempotency and extensivity of F: $F \,; (G \,; F) \sqsubseteq F \,; (F \,; G) = (F \,; F) \,; G = F \,; G \sqsubseteq (F \,; G) \,; F$.

2. Suppose that G is also a closure operator. First notice that monotony

and extensivity are preserved by operator composition. Furthermore, since $G\ ; F \sqsubseteq F\ ; G$, we get $F\ ; G\ ; F\ ; G \sqsubseteq F\ ; F\ ; G\ ; G = F\ ; G$, so, using also extensivity, we obtain $F\ ; G$ idempotent. It follows that $F\ ; G$ is a closure operator. Conversely, by extensivity of F we have $1_{\mathcal{P}(\mathcal{D})} \sqsubseteq F$. Now, since $G\ ; F$ is monotone, it follows from Proposition 4.6 that $G\ ; F \sqsubseteq F\ ; G\ ; F$. But since G is extensive and $F\ ; G$ is idempotent, we further have $F\ ; G\ ; F \sqsubseteq F\ ; G\ ; F\ ; G = F\ ; G$. □

Proposition 4.8 *Consider two classes $\mathcal{C}$ and $\mathcal{D}$, a mapping $\mathcal{U} : \mathcal{C} \to \mathcal{D}$ and two pairs of operators $F = (F_\mathcal{C}, F_\mathcal{D})$ and $G = (G_\mathcal{C}, G_\mathcal{D})$. Then the following hold:*

(1) If $G_\mathcal{C}$ is monotone and $\mathcal{U}$ lifts F and G, then $\mathcal{U}$ also lifts $(F_\mathcal{C}\ ; G_\mathcal{C}, F_\mathcal{D}\ ; G_\mathcal{D})$.
(2) If $F_\mathcal{C}$ and $G_\mathcal{C}$ are extensive, $G_\mathcal{C}$ is monotone and $\mathcal{U}$ preserves fixed points of F and G, then $\mathcal{U}$ also preserves fixed points of $(F_\mathcal{C}\ ; G_\mathcal{C}, F_\mathcal{D}\ ; G_\mathcal{D})$.
(3) If $F_\mathcal{C}$ and $G_\mathcal{C}$ are extensive and $F_\mathcal{C}; G_\mathcal{C}$ is idempotent, then $F_\mathcal{C}; G_\mathcal{C}; F_\mathcal{C} = F_\mathcal{C}; G_\mathcal{C}$.

Proof.

1. We use Proposition 4.6. First take V to be $F_\mathcal{D}$ and U, U' to be $G_\mathcal{D}\ ; \mathcal{U}^{-1}$ and $\mathcal{U}^{-1}; G_\mathcal{C}$ respectively, to obtain $F_\mathcal{D}\ ; G_\mathcal{D}\ ; \mathcal{U}^{-1} \sqsubseteq F_\mathcal{D}\ ; \mathcal{U}^{-1}\ ; G_\mathcal{C}$. Next take V' to be $G_\mathcal{C}$ (which is monotone) and U, U' to be $F_\mathcal{D}\ ; \mathcal{U}^{-1}$ and $\mathcal{U}^{-1}\ ; F_\mathcal{C}$ respectively, to obtain $F_\mathcal{D}\ ; \mathcal{U}^{-1}\ ; G_\mathcal{C} \sqsubseteq \mathcal{U}^{-1}\ ; F_\mathcal{C}\ ; G_\mathcal{C}$. Thus $F_\mathcal{D}\ ; G_\mathcal{D}\ ; \mathcal{U}^{-1} \sqsubseteq \mathcal{U}^{-1}\ ; F_\mathcal{C}\ ; G_\mathcal{C}$.

2. If $\mathcal{C}'$ is a fixed point of $F_\mathcal{C}; G_\mathcal{C}$, then:

- Because $F_\mathcal{C}$ and $G_\mathcal{C}$ are extensive, $\mathcal{C}' \subseteq F_\mathcal{C}(\mathcal{C}') \subseteq G_\mathcal{C}(F_\mathcal{C}(\mathcal{C}'))$; hence $\mathcal{C}' = F_\mathcal{C}(\mathcal{C}')$.
- Because $F_\mathcal{C}$ and $G_\mathcal{C}$ are extensive and $G_\mathcal{C}$ is monotone, $\mathcal{C}' \subseteq G_\mathcal{C}(\mathcal{C}') \subseteq G_\mathcal{C}(F_\mathcal{C}(\mathcal{C}'))$; hence $\mathcal{C}' = G_\mathcal{C}(\mathcal{C}')$.

Thus $\mathcal{C}'$ is a fixed point for $F_\mathcal{C}$ and $G_\mathcal{C}$, making $\mathcal{U}(C')$ a fixed point for $F_\mathcal{D}$ and $G_\mathcal{D}$, therefore a fixed point for $F_\mathcal{D}; G_\mathcal{D}$.

3. $F_\mathcal{C}; G_\mathcal{C} \sqsubseteq F_\mathcal{C}; G_\mathcal{C}; F_\mathcal{C}$ follows from $F_\mathcal{C}$ being extensive. On the other hand, since $G_\mathcal{C}$ is extensive and $F_\mathcal{C}; G_\mathcal{C}$ idempotent, we get $F_\mathcal{C}; G_\mathcal{C}; F_\mathcal{C} \sqsubseteq F_\mathcal{C}; G_\mathcal{C}; F_\mathcal{C}; G_\mathcal{C} = F_\mathcal{C}; G_\mathcal{C}$. □

Proposition 5.2 *For each signature morphism $\phi : \Sigma \to \Sigma'$,*

(1) $Mod(\phi)$ preserves fixed points of P, Fp, Up.
(2) (I) $\Rightarrow$ *$Mod(\phi)$ lifts S, H, Hs and preserves fixed points of Ext* [10].
(3) (IS) *and* (RS) $\Rightarrow$ *$Mod(\phi)$ preserves fixed points of S, Hs, and lifts Ext.*
(4) (IS), (IR) *and* (RS) $\Rightarrow$ *$Mod(\phi)$ preserves fixed points of H.*
(5) (IS) $\Rightarrow$ *$Mod(\phi)$ lifts Ur.*

Proof.

Throughout this proof, for any signature Σ, Σ-morphism $h : A \to B$, and $w = s_1 \dots s_n \in S^*$, $h_w : A^w \to B^w$ denotes the mapping defined by $h_w(a_1, \dots, a_n) = (h_{s_1}(a_1), \dots, h_{s_n}(a_n))$.

1. Follows from the well-known facts that $Mod(\phi)$ preserves canonical direct products and canonical filtered colimits and that canonical filtered products are canonical filtered colimits of canonical direct products.

2. Proved in [10], Proposition 1. Note that for a binary relation R, $Mod(\phi)$ lifts R iff ϕ lifts R^{-1} according to the terminology in [10]; also, our relations S, H and Hs coincide with the inverses of the relations $\stackrel{\mathcal{S}_c}{\to}$, $\stackrel{\mathcal{H}_r}{\leftarrow}$ and $\stackrel{\mathcal{H}_s}{\leftarrow}$ defined in [10], respectively.

3 and 4. Let A' be a Σ'-model and B a Σ-model.

Preservation of fixed points of S and lifting of Ext: Suppose there exists a strong injective morphism $i : B \to A'\restriction_\phi$. Let B' be the following Σ'-model:
- For each $s' \in S'$, let $B'_{s'} = B_s$ if s' has the form $\phi^{st}(s)$ and $B'_{s'} = A'_{s'}$ otherwise. Since ϕ^{st} is injective, the definition is not ambiguous. We can now define for each $s' \in S'$, $i'_{s'} : B'_{s'} \to A'_{s'}$ to be i_s if s' has the form $\phi^{st}(s)$ and $1_{A'_{s'}}$ otherwise;
- For each $\sigma' \in F'_{w' \to s'}$, let $B'_{\sigma'} = B_\sigma$ if σ' has the form $\phi^{op}_{w \to s}(\sigma)$ and $B'_{\sigma'}(\overline{b'}) = A'_{\sigma'}(i'_{w'}(\overline{b'}))$ for each $\overline{b'} \in B'^{w'}$ otherwise. (Note that, because of (RS), in the latter case of the definition $s' \notin \phi^{st}(S)$, thus $A'_{\sigma'}(i'_{w'}(\overline{b'})) \in B_{s'}$.) Let us show that the definition above is not ambiguous. Consider $\sigma_1, \sigma_2 \in F_{w \to s}$ such that $\phi^{op}_{w \to s}(\sigma_1) = \phi^{op}_{w \to s}(\sigma_2)$. Then $(A'\restriction_\phi)_{\sigma_1} = (A'\restriction_\phi)_{\sigma_2}$ and since i is injective it follows that $B_{\sigma_1} = B_{\sigma_2}$.
- For each $\pi' \in P'_{w'}$, let $B'_{\pi'} = (i'_{w'})^{-1}(A'_{\pi'})$.

Thus, B' is a Σ'-model and i' is an injective morphism. Furthermore, i' is strong from the way the relations $B'_{\pi'}$ were defined on B'. Also, the models $B'\restriction_\phi$ and B have the same sort carriers and operations by the definition of B'. Finally, for any $\pi \in P_w$, we have that $B_\pi = (i_w)^{-1}((A'\restriction_\phi)_\pi) = (i'_{\phi^{st}(w)})^{-1}(A'_{\phi^{rl}(\pi)}) = B'_{\phi^{rl}(\pi)} = (B'\restriction_\phi)_\pi$, hence $B'\restriction_\phi$ and B coincide on the relational part too.

Preservation of fixed points of H and Hs: Suppose there exists a surjection $h : A'\restriction_\phi \to B$. Let B' be the following Σ'-model:
- For each $s' \in S'$, let $B'_{s'} = B_s$ if s' has the form $\phi^{st}(s)$ and $B'_{s'} = \{\star\}$ (a singleton) otherwise. Since ϕ^{st} is injective, the definition is not ambiguous. We now define for each $s' \in S'$, $h'_{s'} : A'_{s'} \to B'_{s'}$ to be h_s if s' has the form $\phi^{st}(s)$ and the only possible mapping otherwise;
- For each $\sigma' \in F'_{w' \to s'}$, let $B'_{\sigma'} = B_\sigma$ if σ' has the form $\phi^{op}_{w \to s}(\sigma)$ and $B'_{\sigma'}(\overline{b'}) = \star$ for each $\overline{b'} \in B'_{w'}$ otherwise. (Note that, because of (RS), in the latter case

of the definition s' does not have the form $\phi^{st}(s)$, thus $B'_{s'} = \{\star\}$.) Let us show the definition above is not ambiguous. Consider $\sigma_1, \sigma_2 \in F_{w \to s}$ such that $\phi^{op}_{w \to s}(\sigma_1) = \phi^{op}_{w \to s}(\sigma_2)$. Then $(A'\lceil_\phi)_{\sigma_1} = (A'\lceil_\phi)_{\sigma_2}$ and, since h is surjective, it follows that $B_{\sigma_1} = B_{\sigma_2}$.
- Let $\pi' \in P'_{w'}$. If h is strong, let $B'_{\pi'} = h'_{w'}(A'_{\pi'})$. If h is not strong (thus we work under the hypothesis that ϕ^{rl} is injective), let $B'_{\pi'} = B_\pi$ if π' has the form $\phi^{rl}(\pi)$ and $B'_{\pi'} = B'^{w'}$ otherwise.

Thus, B' is a Σ'-model and h' is a surjective morphism. Furthermore, the models $B'\lceil_\phi$ and B have the same sort carriers and operations by the definition of B'. If h is not strong, then $B'\lceil_\phi$ and B coincide on the relational part too, by the definition of B'. On the other hand, if h is strong, then for any $\pi \in P_w$ we have that $B'_{\pi'} = h_{w'}(A'_{\pi'})$, hence $B_\pi = h_w((A'\lceil_\phi)_\pi) = h'_{\phi^{st}(w)}(A'_{\phi^{rl}(\pi)}) = B'_{\phi^{rl}(\pi)} = (B'\lceil_\phi)_\pi$. Note that in case h is strong, h' is strong too.

5. Let A' be a Σ'-model and let B be a Σ-model isomorphic to an ultrapower of $A'\lceil_\phi$, say $A'\lceil_\phi{}^I/_\mathcal{F}$. Let $C' = A'^I/_\mathcal{F}$. It is known [7] that $C'\lceil_\phi$ is equal to $A'\lceil_\phi{}^I/_\mathcal{F}$, hence is isomorphic to B. Since ϕ has (IS), it is easy to define a Σ'-model B' such that $B'\lceil_\phi = B$ and B' is isomorphic to C', whence $B' \, Ur \, A'$. $\square$