A Concrete Introduction to Abstract Coinductive Datatypes

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This is continuation of

`www.andreipopescu.uk/resourcesForStudents/introductionToDatatypes.pdf`

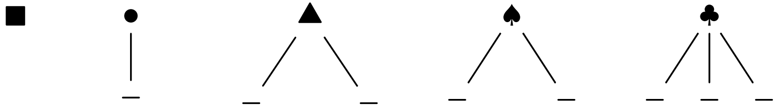
See also

`www.andreipopescu.uk/resourcesForStudents/codatatypesInIsabelleHOL.pdf`

`www.andreipopescu.uk/slides/ESOP2015-slides.pdf`

Recall: It's All About Shape and Content

Shapes



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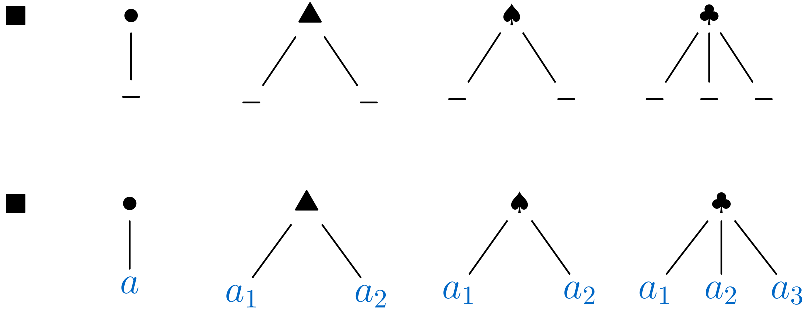

Shapes filled with content from a set $A = \{a_1, a_2, \ldots\}$

Recall: Natural Functors on Set

Set = the class of all sets

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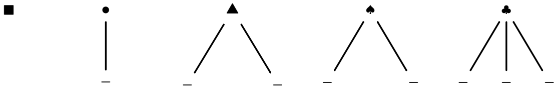

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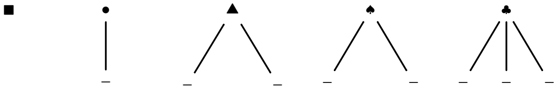

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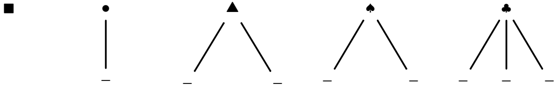

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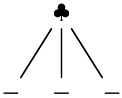

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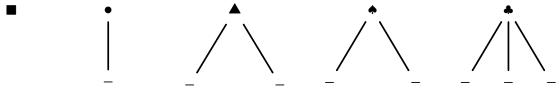

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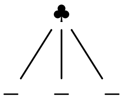

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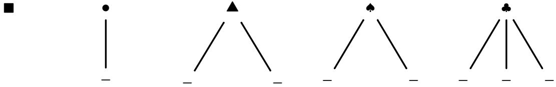

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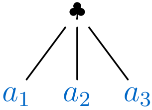

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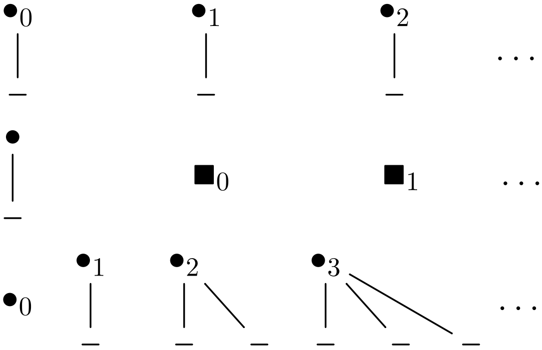
Recall: Examples of Natural Functors

$\mathsf{F}\ A = \mathbb{N} \times A$

$\mathsf{F}\ A = \mathbb{N} + A$

$\mathsf{F}\ A = \mathsf{List}\ A$



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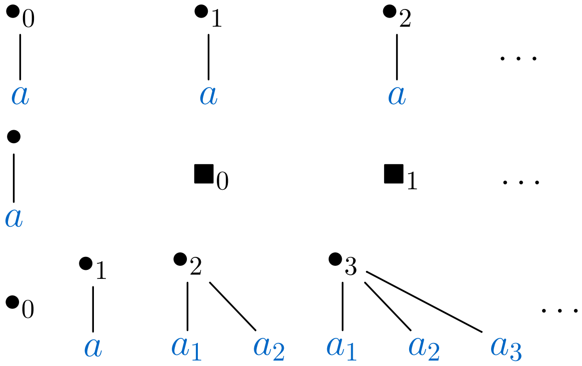

Functorial Action (Mapper)

$$\begin{array}{c} A \\ \Big\downarrow f \\ B \end{array}$$

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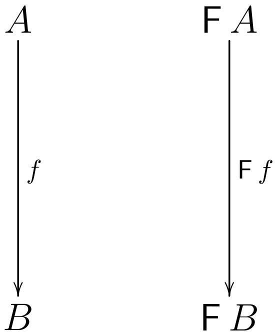

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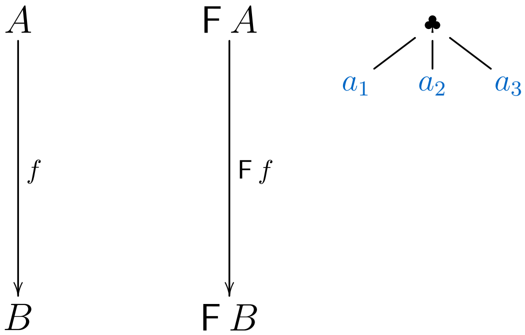

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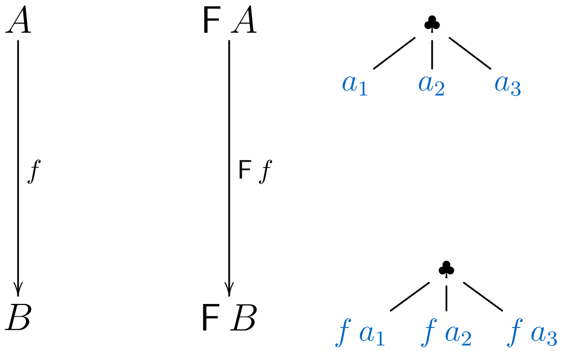

Keep the same shape

Apply f to the content

Atoms

$$\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$$

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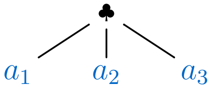

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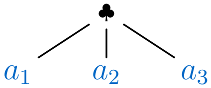

$$\{a_1, a_2, a_3\}$$

$$\mathsf{F}\,A \xrightarrow{\mathsf{Fatoms}_A} \mathcal{P}\,A$$

Natural Functors

$\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

Functoriality: For all $A \xrightarrow{f} B$, we have $\mathsf{F}\,A \overset{\mathsf{F}\,f}{\Rightarrow} \mathsf{F}\,B$ such that:

$$\mathsf{F}\ \mathsf{id}_A = \mathsf{id}_{\mathsf{F}\,A}$$
$$\mathsf{F}\,(g \circ f) = \mathsf{F}\,g \circ \mathsf{F}\,f$$

Naturality: For all A, we have $\mathsf{F}\,A \overset{\mathsf{Fatoms}_A}{\Rightarrow} \mathcal{P}\,A$ such that, for all $A \xrightarrow{f} B$:

$$\mathsf{image}\ f \circ \mathsf{Fatoms}_A = \mathsf{Fatoms}_B \circ \mathsf{image}\ f$$

Examples

$$A \overset{f}{\Rightarrow} B \qquad \mathsf{F}\ A \overset{\mathsf{F}\,f}{\Rightarrow} \mathsf{F}\ B \qquad \mathsf{F}\ A \overset{\mathsf{Fatoms}}{\Rightarrow} \mathcal{P}\ A$$

$\mathsf{F}\ A = \mathbb{N} \times A$

$$\mathsf{F} f\ (n, a) = (n, f\ a)$$
$$\mathsf{Fatoms}\ (n, a) = \{a\}$$

$\mathsf{F}\ A = \mathbb{N} + A$

$$\mathsf{F} f\ (\mathsf{Left}\ n) = \mathsf{Left}\ n \qquad \mathsf{F} f\ (\mathsf{Right}\ a) = \mathsf{Right}\ (f\ a)$$
$$\mathsf{Fatoms}\ (\mathsf{Left}\ n) = \varnothing \qquad \mathsf{Fatoms}\ (\mathsf{Right}\ a) = \{a\}$$

$\mathsf{F}\ A = \mathsf{List}\ A$

$$\mathsf{F} f\ (a_1 \cdot a_2 \cdot \ldots \cdot a_n) = f\ a_1 \cdot f\ a_2 \ldots \cdot f\ a_n$$
$$\mathsf{Fatoms}\ (a_1 \cdot a_2 \cdot \ldots \cdot a_n) = \{a_1, a_2, \ldots, a_n\}$$

Recall: Iterating Shape Composition

Natural functor $\mathsf{F} : \mathsf{Set} \to \mathsf{Set}$

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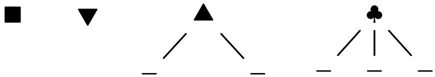
The shapes of F:



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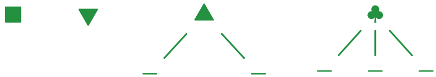

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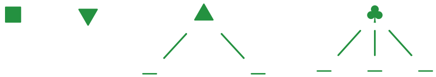

Put them together by plugging in shape for content slot

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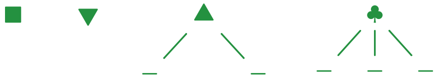
Copies of the shapes of F: 

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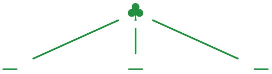

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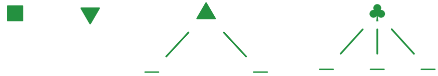

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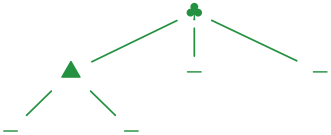

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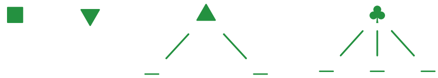

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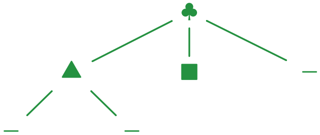

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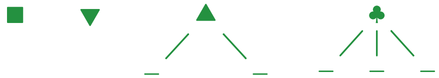

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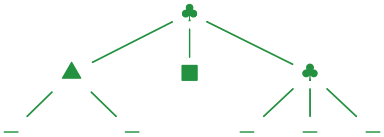

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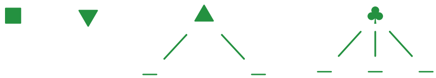

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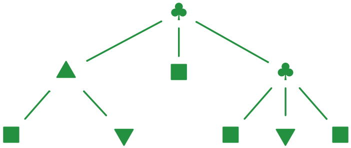

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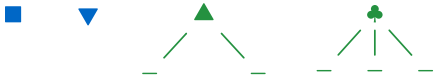

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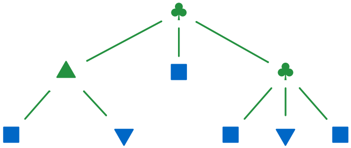

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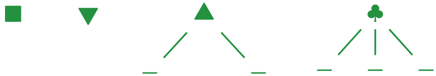

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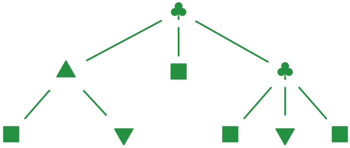


Define I_F = the set of all such finitary couplings

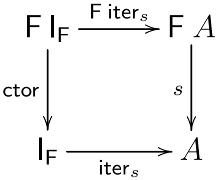
Recall: Properties of $\mathsf{I_F}$

Given a natural functor F, $(\mathsf{I_F},\ \mathsf{ctor} : \mathsf{F}\ \mathsf{I_F} \to \mathsf{I_F})$ satisfies:

ctor bijection

$\mathsf{I_F}$ = the datatype of F

Iteration (Initial Algebra Property): For all $(A, s : \mathsf{F}\ A \to A)$, there exists a unique function iter_s such that



Induction: Given any predicate φ on $\mathsf{I_F}$

$$\frac{\forall x \in \mathsf{F}\ \mathsf{I_F}.\ (\forall \mathsf{i} \in \mathsf{Fatoms}\ \mathsf{x}.\ \varphi\ \mathsf{i}) \Rightarrow \varphi\ (\mathsf{ctor}\ \mathsf{x})}{\forall i \in \mathsf{I_F}.\ \varphi\ \mathsf{i}}$$

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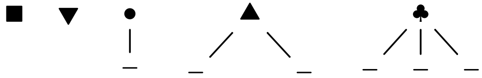

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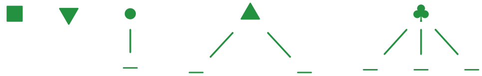

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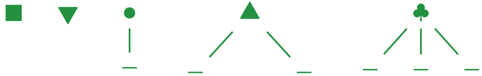

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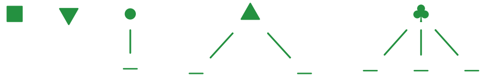

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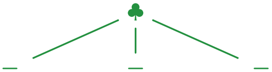

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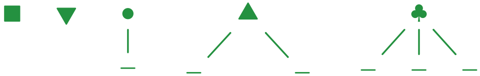
Copies of the shapes of F: 

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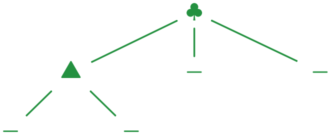

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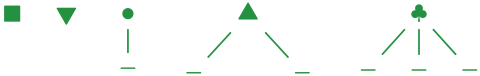
Copies of the shapes of F: 

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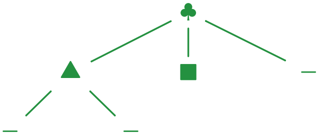

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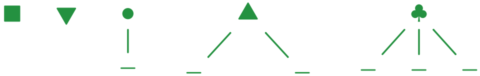
Copies of the shapes of F: 

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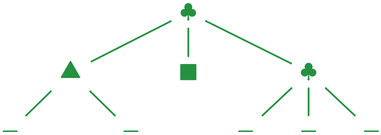

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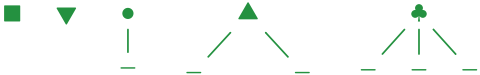

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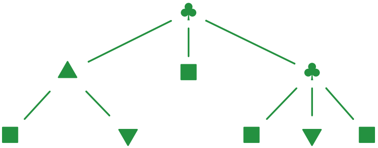

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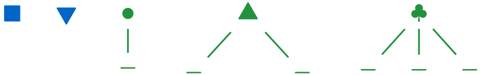

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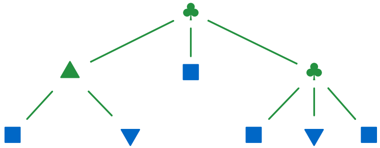

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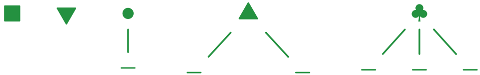
Copies of the shapes of F: 

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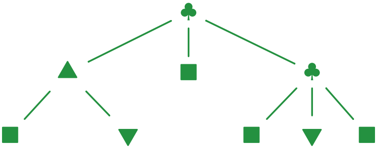

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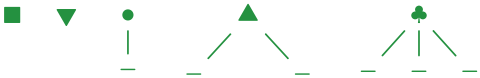
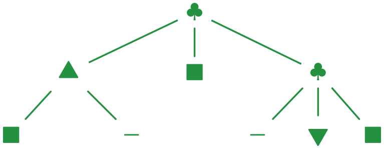
Copies of the shapes of F: 

Put them together by plugging in shape for content slot
until there are no lingering slots left!



Allow infinite couplings

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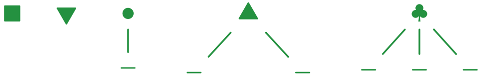
Copies of the shapes of F: 

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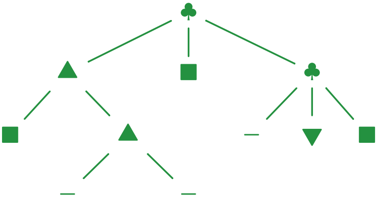

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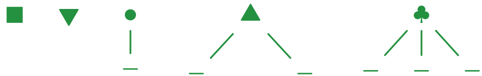

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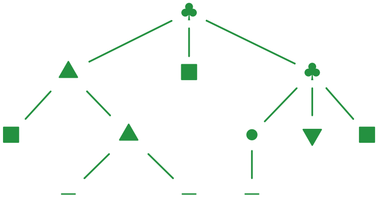

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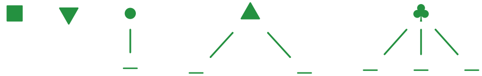

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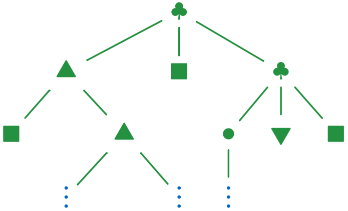

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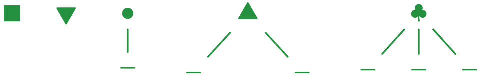

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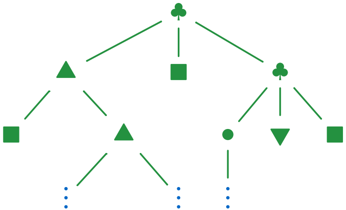
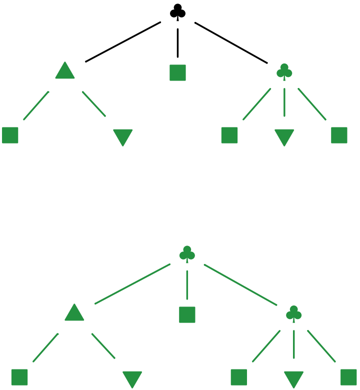
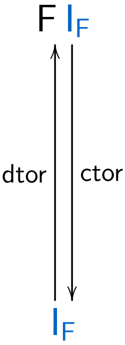


Define J_F = the set of all such (possibly) infinitary couplings

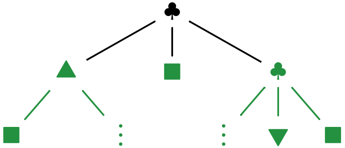
Recall: Properties of I_F: Bijectivity

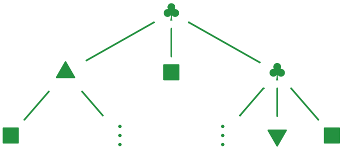


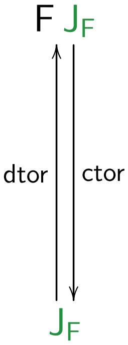


ctor and dtor are mutually inverse bijections

Properties of J_F: Bijectivity







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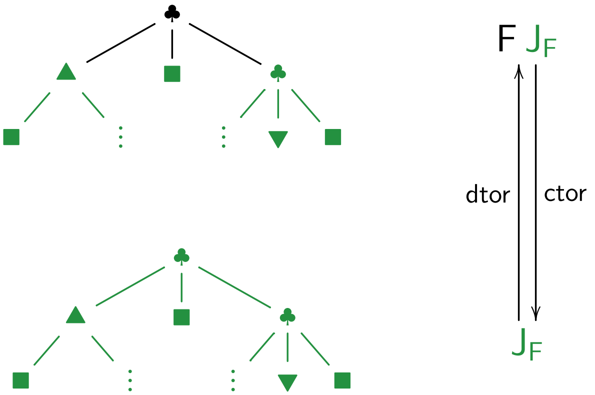
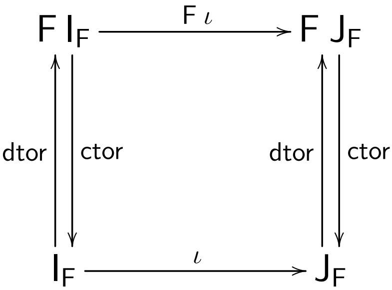
Properties of J_F: Bijectivity



ctor and dtor are mutually inverse bijections

A similar property holds for J_F, where we use the same notations for constructor and destructor

I_F is embedded in J_F



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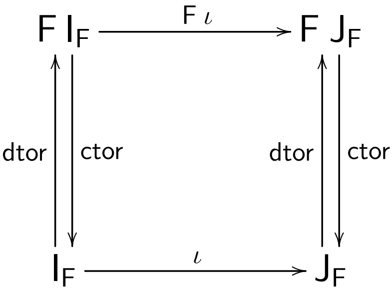
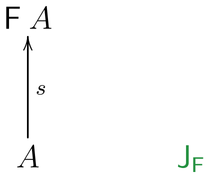


$\iota = \text{iter}_{\text{ctor}:F\ J_F \to F\ J_F}$

Properties of $\mathsf{J_F}$: Coiteration



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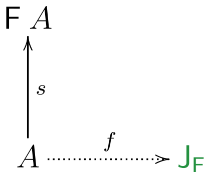

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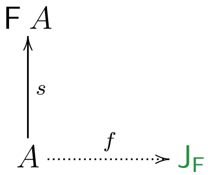

a

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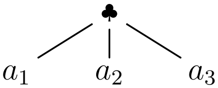

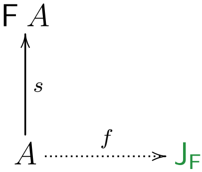

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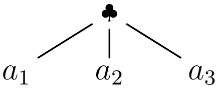

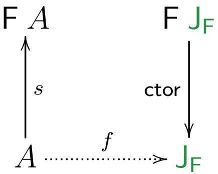

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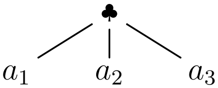

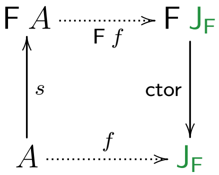

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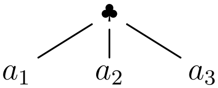

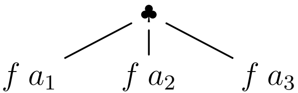

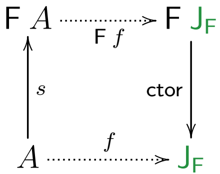

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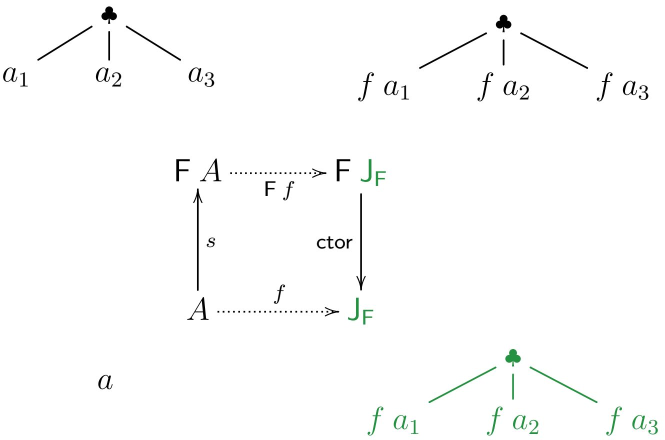

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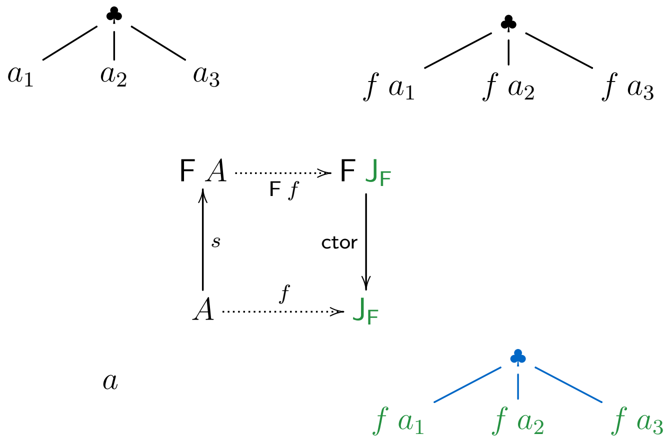
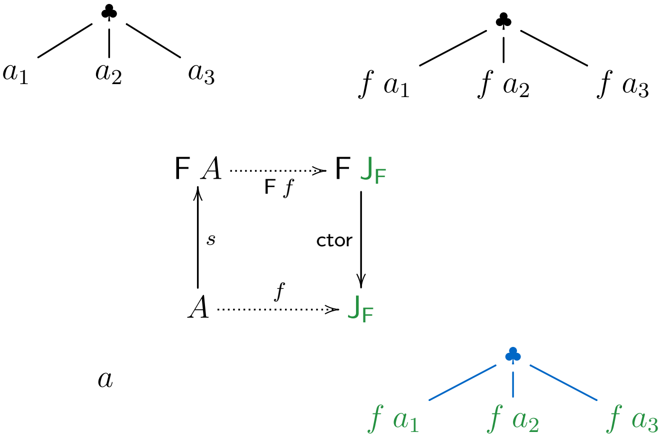


a_1, a_2, a_3 are not “smaller” than a in any sense

Properties of $\mathsf{J_F}$: Coiteration



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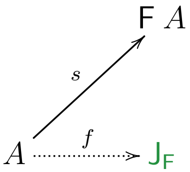
But computation has made progress

Properties of $\mathsf{J_F}$: Coiteration



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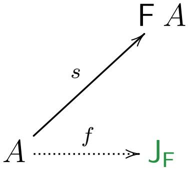

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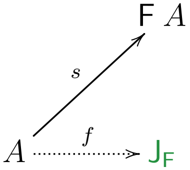
Properties of $\mathsf{J_F}$: Coiteration

$s\ a$



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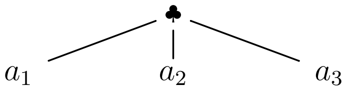

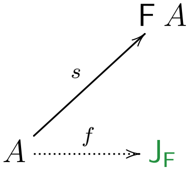

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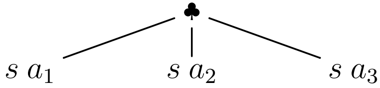

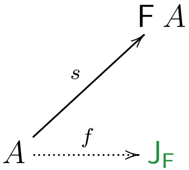

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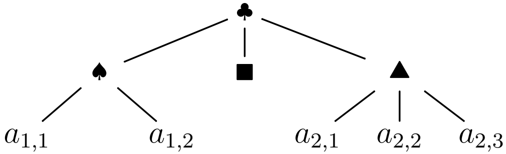

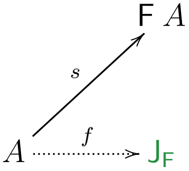

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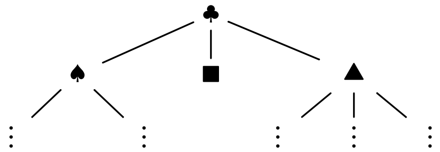

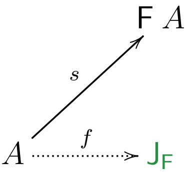

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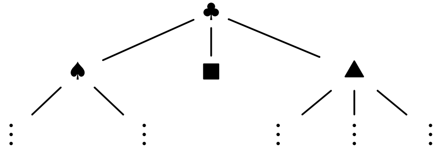

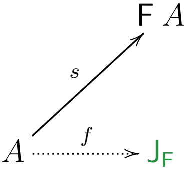

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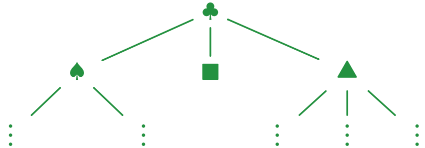

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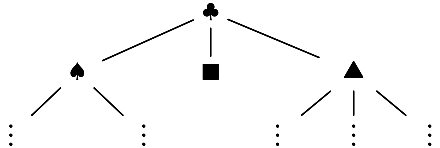
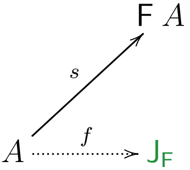
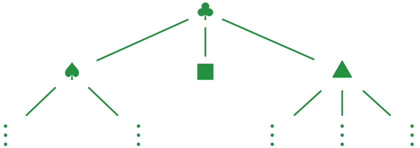




$s\ a$ = the seed encoding the growth of the tree $f\ a$

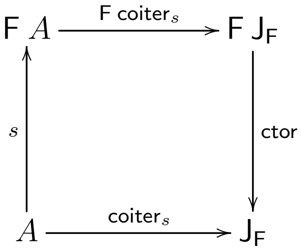
a



Properties of $\mathsf{J_F}$: Coiteration

Given a natural functor F, $(\mathsf{J_F},\ \mathsf{dtor} : \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F})$

Coiteration (Final Coalgebra Property): For all $(A, s : A \to \mathsf{F}\ A)$, there exists a unique function coiter_s with



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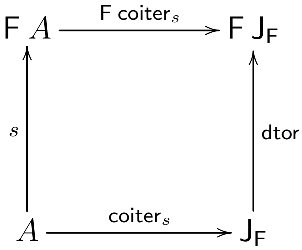

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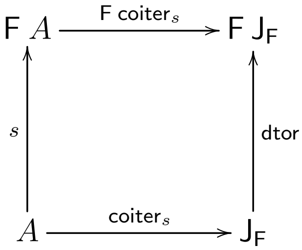
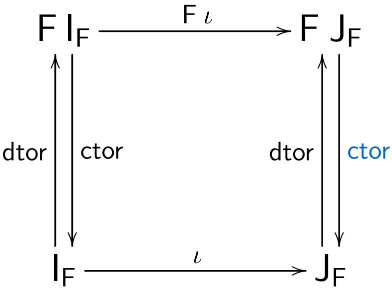


$\mathsf{J_F}$ = the codatatype of F

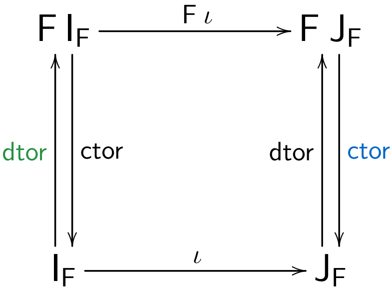
The I_F to J_F embedding revisited



ι can be regarded as defined by iteration on I_F

$\iota = \mathrm{iter}_{\mathrm{ctor}}$

The I_F to J_F embedding revisited



ι can be regarded as defined by
iteration on I_F but also by coiteration on J_F!

$\iota = \text{iter}_{\text{ctor}} = \text{coiter}_{\text{dtor}}$

Properties of J_F: Coinduction

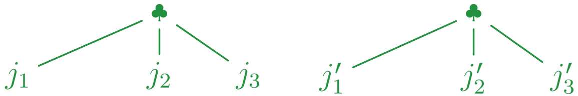
j j'

Properties of $\mathsf{J_F}$: Coinduction

j $\qquad$ j'

Want: $\quad j = j'$

Properties of J_F: Coinduction



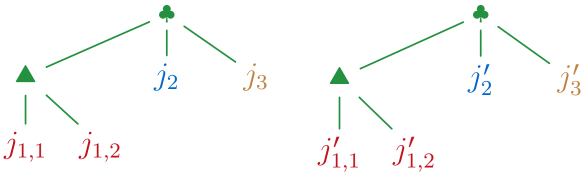
Want: $j = j'$

Properties of J_F: Coinduction

Suffices:

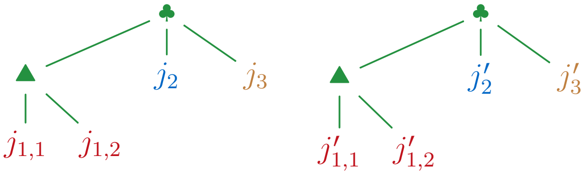
$$j_1 = j_1'$$
$$j_2 = j_2'$$
$$j_3 = j_3'$$

Properties of J_F: Coinduction



Suffices: $j_1 = j'_1$
$j_2 = j'_2$
$j_3 = j'_3$

Properties of J_F: Coinduction

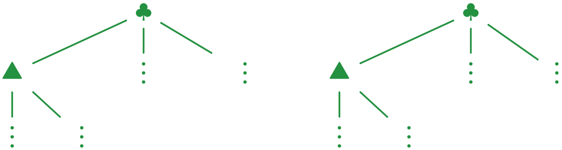


Suffices: $j_{1,1} = j'_{1,1}, \; j_{1,2} = j'_{1,2}$

$$j_2 = j'_2$$

$$j_3 = j'_3$$

Properties of $\mathsf{J_F}$: Coinduction



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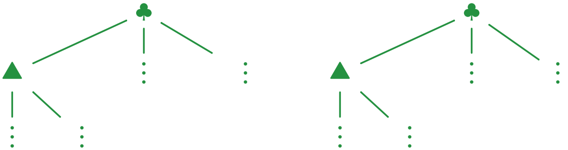
Properties of J_F: Coinduction



If we can stay in the game indefinitely, then equality holds!

Suffices: $j_{1,1} = j'_{1,1},\ j_{1,2} = j'_{1,2}$

$$j_2 = j'_2$$

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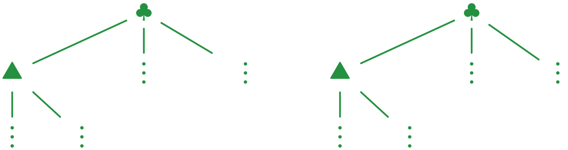
Properties of J_F: Coinduction



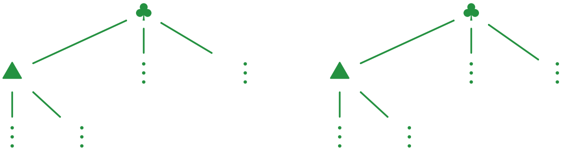
If we can stay in the game indefinitely, then equality holds!
But how to show we can "stay in the game"?

Suffices: $j_{1,1} = j'_{1,1},\ j_{1,2} = j'_{1,2}$

$$j_2 = j'_2$$

$$j_3 = j'_3$$

Properties of J_F: Coinduction



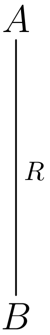
If we can stay in the game indefinitely, then equality holds!
But how to show we can "stay in the game"?
By exhibiting a "strategy"

Suffices: $j_{1,1} = j'_{1,1},\ j_{1,2} = j'_{1,2}$

$$j_2 = j'_2$$

$$j_3 = j'_3$$

But First: Relators



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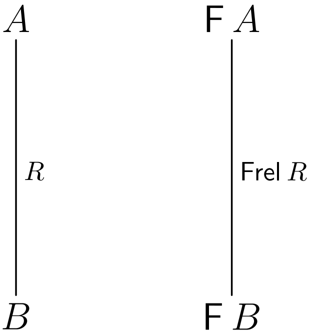

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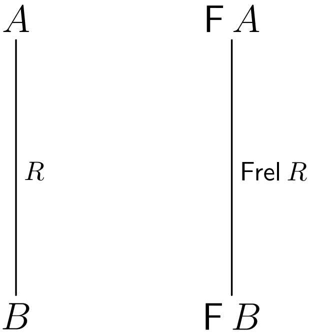

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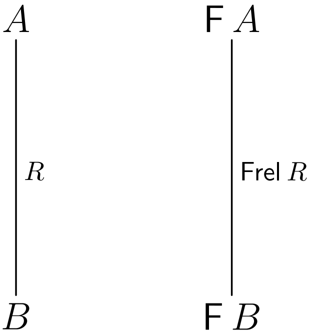

Two elements of F A and F B are related by Frel R iff

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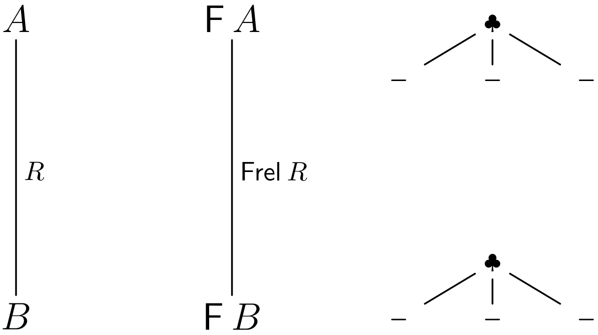

Two elements of F A and F B are related by Frel R iff they have the same shape

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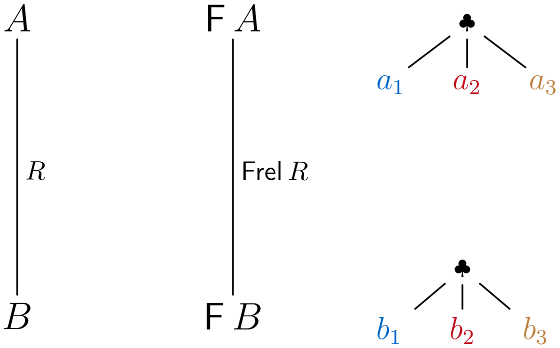

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and the contents from corresponding slots are related by R

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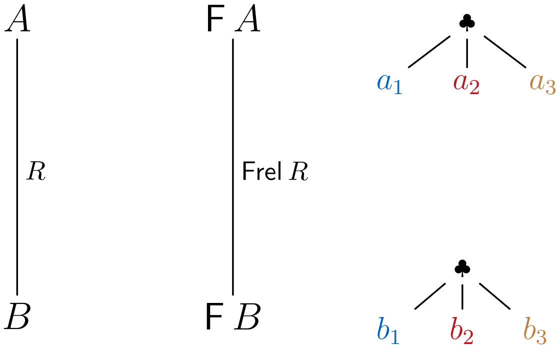

Two elements of F A and F B are related by Frel R iff
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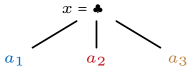
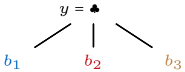
$R\ a_1\ b_1$, $R\ a_2\ b_2$, $R\ a_3\ b_3$

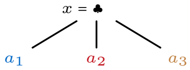
Relator Defined from Mapper

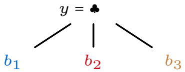




R relation between A and B, $x \in \mathsf{F}\ A$, $y \in \mathsf{F}\ B$

Relator Defined from Mapper





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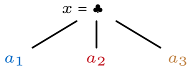

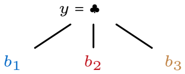

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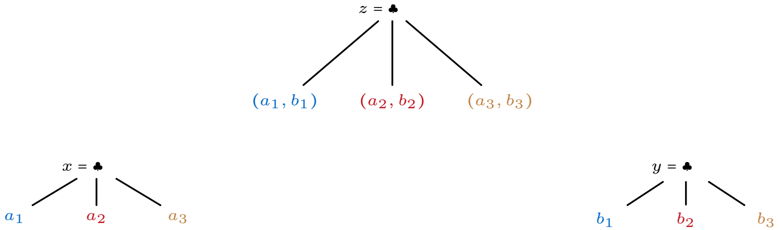

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Relators for the Running Examples

R relation between A and B

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$\mathsf{Frel}\ R$ relation between $\mathsf{F}\ A$ and $\mathsf{F}\ B$

$$\mathsf{F}\ A = \mathbb{N} \times A \qquad \mathsf{Frel}\ R\ (m, a)\ (n, b) \Leftrightarrow (m = n\ \wedge\ R\ a\ b)$$

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$$\begin{aligned}&\mathsf{Frel}\ R\ u\ v \Leftrightarrow \\ &\quad (\exists n.\ u = v = \mathsf{Left}\ n) \vee \\ &\quad (\exists a, b.\ u = \mathsf{Right}\ a \ \wedge\ v = \mathsf{Right}\ b \ \wedge\ R\ a\ b)\end{aligned}$$

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F A = List A

Relators for the Running Examples

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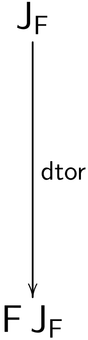
Back to the “Strategy” for Proving Equality



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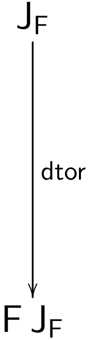
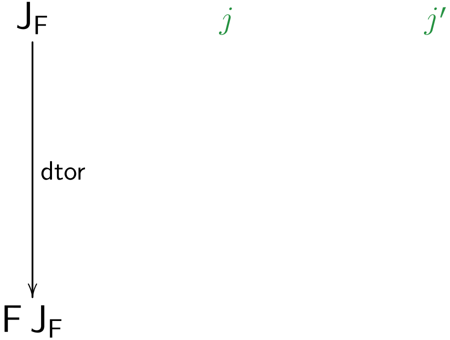


Given binary relation R on J_F

Back to the “Strategy” for Proving Equality



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If $\forall j, j'.\ R\ j\ j'$

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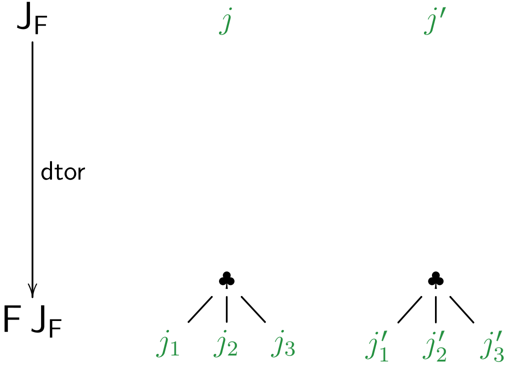
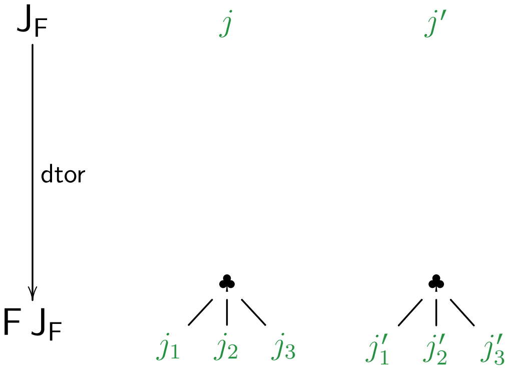


Given binary relation R on $\mathsf{J_F}$

If $\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')$

Back to the “Strategy” for Proving Equality



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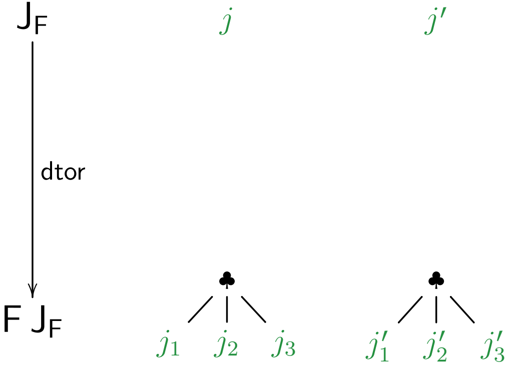
Then R is included in equality

Back to the “Strategy” for Proving Equality



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If $\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')$
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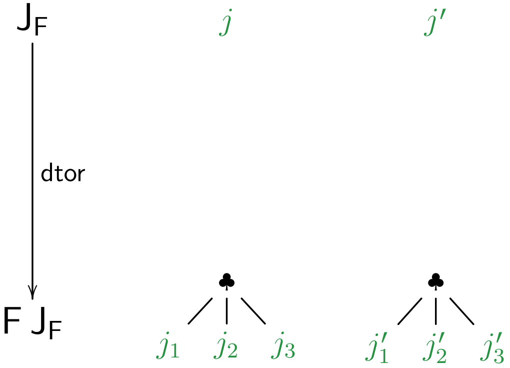
Back to the "Strategy" for Proving Equality



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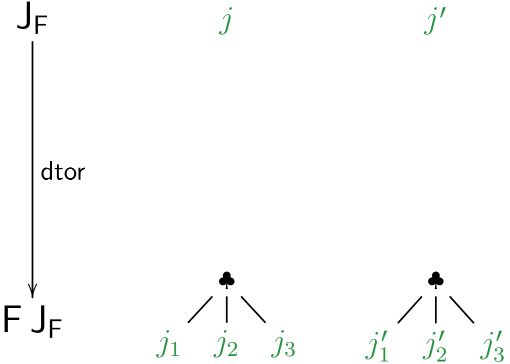


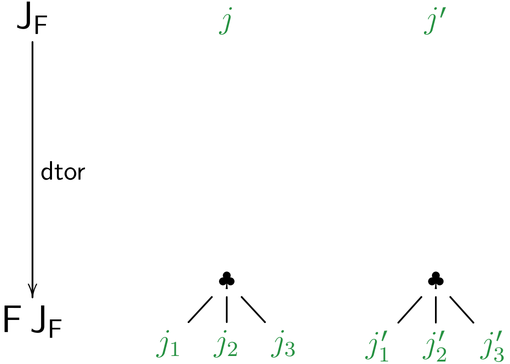
Summary: to prove $j = j'$,

Given binary relation R on J_F

If $\forall j, j'.\ R\ j\ j' \Rightarrow \text{Frel}\ R\ (\text{dtor}\ j)\ (\text{dtor}\ j')$ $\boxed{R \text{ F-bisimulation}}$

Then R is included in equality $\quad \forall j, j'.\ R\ j\ j' \Rightarrow j = j'$

Back to the “Strategy” for Proving Equality



Summary: to prove $j = j'$, find F-bisimulation R with $R\ j\ j'$

Given binary relation R on $\mathsf{J_F}$

If $\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')$ R F-bisimulation

Then R is included in equality $\forall j, j'.\ R\ j\ j' \Rightarrow j = j'$

Summary for J_F

Given a natural functor F, $(J_F,\ \text{dtor} : J_F \to F\ J_F)$ satisfies:

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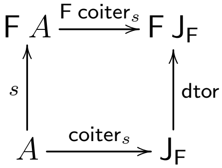
Coiteration (Final Coalgebra Property): For all $(A, s : A \to \mathsf{F}\ A)$, there exists a unique function coiter_s with



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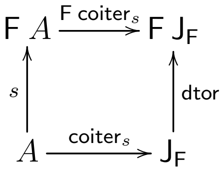

Coinduction: Given any binary relation R on J_F

$$\frac{R \text{ is an F-bisimulation}}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

Summary for $\mathsf{J_F}$

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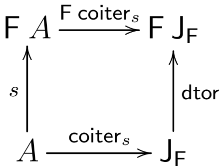

Coinduction: Given any binary relation R on $\mathsf{J_F}$

$$\frac{\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

Summary for $\mathsf{J_F}$

Given a natural functor F, $\quad (\mathsf{J_F},\ \mathsf{dtor} : \mathsf{J_F} \to \mathsf{F}\ \mathsf{J_F})$ satisfies:

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$\mathsf{J_F}$ = the codatatype of F

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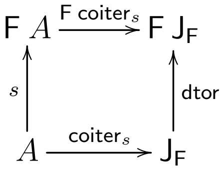

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Example of Codatatype

Let B be a fixed set. $\mathsf{F}\ A = B \times A$

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Who is J_{F}?

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Who is $\mathsf{J_F}$?

Its elements have the form $(b_1, (b_2, \ldots, (b_n, \ldots$

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I.e., essentially streams $b_1 \cdot b_2 \cdot \ldots \cdot b_n \cdot \ldots$

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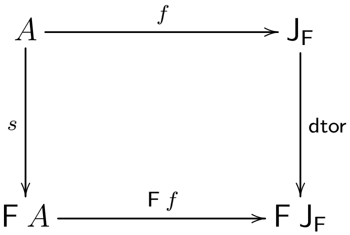
Its elements have the form $(b_1, (b_2, \ldots, (b_n, \ldots$
I.e., essentially streams $b_1 \cdot b_2 \cdot \ldots \cdot b_n \cdot \ldots$
So $\mathsf{J_F} = \mathsf{Stream_B}$

Example of Codatatype: Stream

B fixed $\mathsf{F}\ A = B \times A$ $f = \mathsf{coiter}_s$ $\mathsf{J_F} = \mathsf{Stream_B}$



$\mathsf{dtor}\ (f\ a) = (\mathsf{F}\ f)\ (s\ a)$

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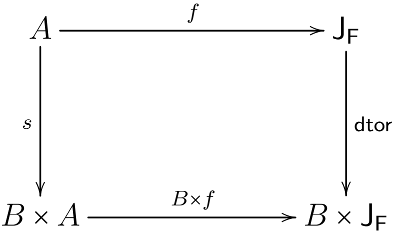

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Define: $\mathsf{hd} = \pi_1 \circ \mathsf{dtor}$ $\mathsf{tl} = \pi_2 \circ \mathsf{dtor}$
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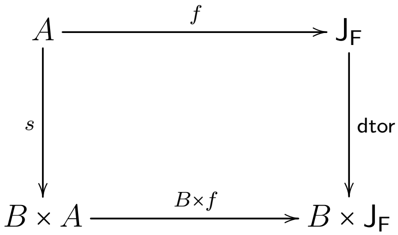

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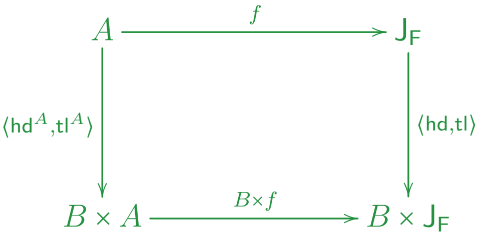

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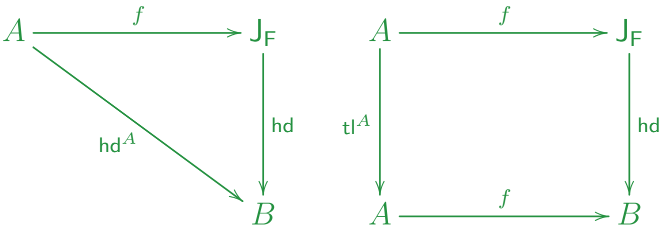

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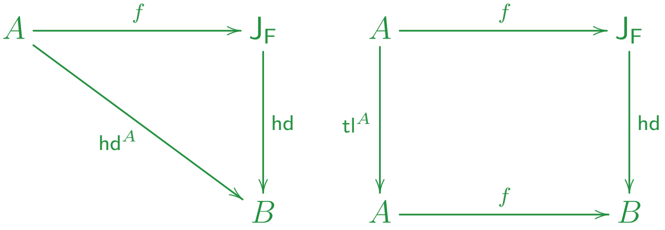


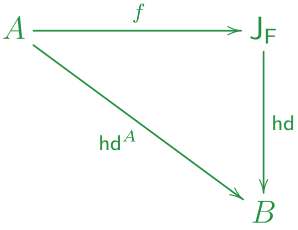
$\mathsf{hd}\ (f\ a) = \mathsf{hd}^A\ a$

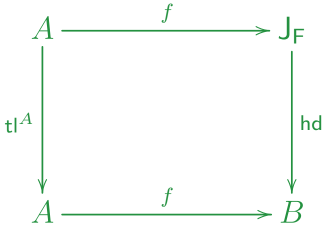
$\mathsf{tl}\ (f\ a) = f\ (\mathsf{tl}^A\ a)$

Example of Codatatype: Stream

B fixed $\mathsf{F}\, A = B \times A$ $f = \mathsf{coiter}_s$ $\mathsf{J_F} = \mathsf{Stream_B}$

Define: $\mathsf{hd} = \pi_1 \circ \mathsf{dtor}$ $\mathsf{tl} = \pi_2 \circ \mathsf{dtor}$
$\mathsf{hd}^A = \pi_1 \circ s$ $\mathsf{tl}^A = \pi_2 \circ s$





$\mathsf{hd}\,(f\ a) = \mathsf{hd}^A\ a$
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Standard stream coiteration

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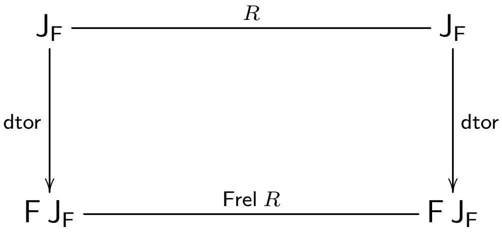

$$\frac{R \text{ is an F-bisimulation}}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

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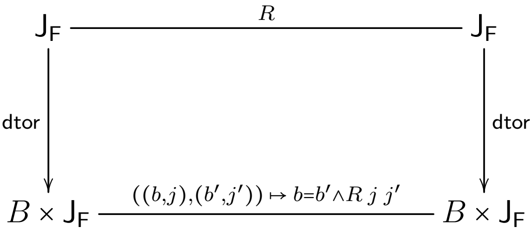

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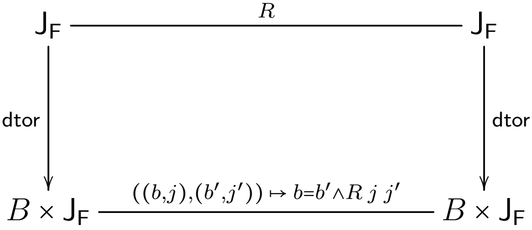

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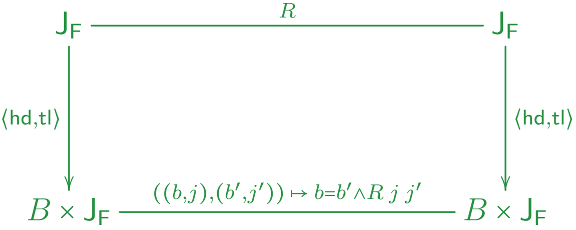

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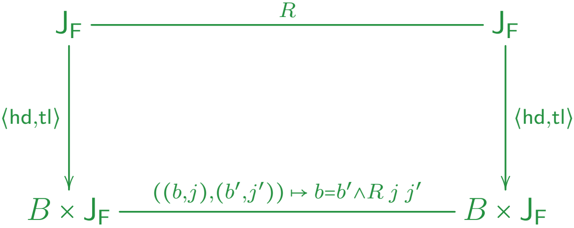

$$\frac{\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{dtor}\ j)\ (\mathsf{dtor}\ j')}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

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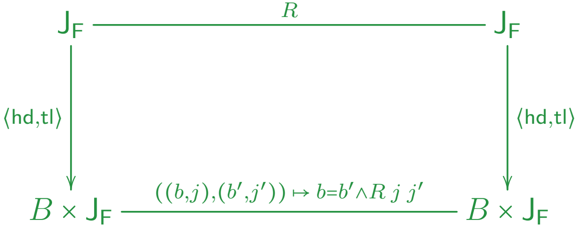

$$\frac{\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{Frel}\ R\ (\mathsf{hd}\ j, \mathsf{tl}\ j)\ (\mathsf{hd}\ j', \mathsf{tl}\ j')}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

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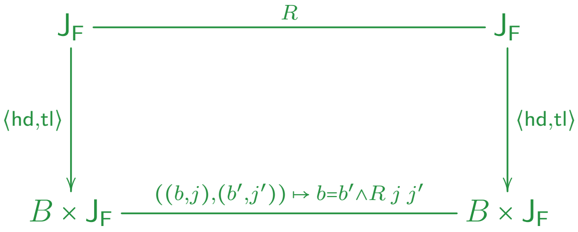

$$\frac{\forall j, j'.\ R\ j\ j' \Rightarrow \mathsf{hd}\ j = \mathsf{hd}\ j' \ \wedge\ R\ (\mathsf{tl}\ j)\ (\mathsf{tl}\ j')}{\forall j, j'.\ R\ j\ j' \Rightarrow j = j'}$$

Concrete Example of Coiteration

$$\begin{aligned} &\mathsf{ev} : \mathsf{Stream}_B \to \mathsf{Stream}_B \\ &\quad \mathsf{hd}\ (\mathsf{ev}\ j) = \mathsf{hd}\ j \\ &\quad \mathsf{tl}\ (\mathsf{ev}\ j) = \mathsf{ev}\ (\mathsf{tl}\ (\mathsf{tl}\ j)) \end{aligned}$$

Concrete Example of Coiteration

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$\mathsf{odd} : \mathsf{Stream}_B \to \mathsf{Stream}_B$

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$\quad \mathsf{tl}\,(\mathsf{odd}\ j) = \mathsf{odd}\,(\mathsf{tl}\,(\mathsf{tl}\ j))$

$\mathsf{zip} : \mathsf{Stream}_B \times \mathsf{Stream}_B \to \mathsf{Stream}_B$

$\quad \mathsf{hd}\,(\mathsf{zip}\,(j_1, j_2)) = \mathsf{hd}\ j_1$

$\quad \mathsf{tl}\,(\mathsf{zip}\,(j_1, j_2)) = \mathsf{zip}\,(j_2,\ \mathsf{tl}\ j_1)$

Pattern-Based Incremental Coinduction

$\mathsf{zip}\,(\mathsf{ev}\ j, \mathsf{odd}\ j) = j$

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$\mathsf{hd}\ \ldots = \mathsf{hd}\,(\mathsf{tl}\ j)$

Bisimulation: $R\ j_1\ j_2 \equiv$
$j_1 = \mathsf{zip}\,(\mathsf{ev}\ j_2, \mathsf{odd}\ j_2) \lor$
$\exists j.\ j_1 = \mathsf{zip}\,(\mathsf{odd}\ j,\ \mathsf{ev}\,(\mathsf{tl}\,(\mathsf{tl}\ j))) \land j_2 = \mathsf{tl}\ j$

Universe of (Co)Datatypes

Natural functors are a class of functors

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Nesting datatypes in codatatypes or vice versa
allows for modular specs of fancy data structures

Universe of (Co)Datatypes in Isabelle/HOL

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In the background:

- Isabelle parses this into a natural functor: $B \mapsto B \times A$
- Then infers high-level principles for (co)recursion and (co)induction for Stream
- Finally, Stream is itself registered as a natural functor

Examples

datatype List A = Nil $|$ Cons A (List A)

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

Examples

datatype List A = Nil $|$ Cons A (List A)

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datatype X A = Leaf A $|$ Node (X A) (X A)

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node $(X\ A)\ (X\ A)$

Examples

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datatype BTree A = Leaf A $|$ Node $(X\ A)$ $(X\ A)$

datatype $X\ A$ = Node A (List $(X\ A)$)

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

datatype Tree A = Node A (List (Tree A))

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finite-depths, finitely branching
A-labeled trees

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A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (Lazy_List (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (Countable_Set (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (Set$_k$ (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (Multi_Set (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (Fuzzy_Set (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (PLUG_YOUR_OWN (Tree A))
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Examples

datatype List A = Nil | Cons A (List A)

codatatype Lazy_List A = Nil | Cons A (List A)

datatype BTree A = Leaf A | Node (X A) (X A)

codatatype Tree A = Node A (PLUG_YOUR_OWN (Tree A))
infinite-depths, infinitely branching unordered
A-labeled trees

- Show a set operator to be a (bounded) natural functor
- Register it
- Then Isabelle will allow nesting it in (co)datatype expressions

Examples

datatype X A =
    Elements (Finite_Set (X A))

Examples

datatype Hereditarily_Finite_Set A =
  Elements (Finite_Set (Hereditarily_Finite_Set A))

Examples

datatype Hereditarily_Finite_Set A =
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... in the presence of the Foundation Axiom

Examples

codatatype Hereditarily_Finite_Set A =
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... in the presence of Aczel's Anti-Foundation Axiom

Summary

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Moreover, the abstract constructions have very concrete intuitions

The abstract reality can be very concrete

Relevant Literature

Our work:

Natural functors:
LICS'12, ESOP'15
Flexible corecursion:
ICFP'15
Isabelle implementation:
ITP'14
Case study:
IJCAR'14,
Incremental coinduction:
FOSSACS'10
Non-free datatypes:
LICS'10, ICFP'11, CPP'13

Other people's work:

Isabelle/ZF codata (Paulson)
Containers
(Abbott, Altenkirch, Ghani)
Fibrations
(Hermida, Jacobs)
Flexible Corecursion
(Turi/Plotkin, Bartels, Jacobs,
Milius, Hinze, Atkey/McBride)
Flexible Coinduction
(Rot, Bonsangue, Rutten, Silva,
Roșu, Endrullis, Hendriks,
Hur/Dreyer/Vafeiadis)
Corecursion in MiniAgda, Agda
(Abel)